Q-4) Let r be the radius of convergence for the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n-1)}\, x^n$.

a) Find r. *(10 points)*

b) Check the convergence of the series for $x = r$ and for $x = -r$. *(5+5 points)*

Solution: Let $a_n(x) = \dfrac{n!}{2 \cdot 5 \cdot 8 \cdots (3n-1)}\, x^n$. We use the ratio test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \lim_{n\to\infty} \frac{n+1}{3n+2}\, |x| = \frac{|x|}{3} < 1,$$

gives $|x| < 3$, so the radius of convergence is 3.

To check the end points, we observe that

$$\left| \frac{a_{n+1}(\pm 3)}{a_n(\pm 3)} \right| = \frac{3n+3}{3n+2} > 1,$$

so in particular we have

$$|a_{n+1}(\pm 3)| > |a_n(\pm 3)| > \cdots > |a_1(\pm 4)| = \frac{3}{2}.$$

Hence the general term does not converge to zero as n goes to infinity. Thus the series does not converge at the end points.