Q-5) Find

$$\lim_{x\to 0} \frac{(\tan^{-1} x)(\ln(1+x^2)) - x^3}{(\sec x - 1)(\sin x - x)}.$$

Solution: Here we use the following formulas given on the cover page of the exam booklet:

$$\begin{aligned}
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\
\sec(x) &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots \\
\tan^{-1}(x) &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\end{aligned}$$

Now we have

$$\begin{aligned}
\lim_{x\to 0} \frac{(\tan^{-1} x)(\ln(1+x^2)) - x^3}{(\sec x - 1)(\sin x - x)} &= \lim_{x\to 0} \frac{\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots\right)\left(x^2 - \frac{x^4}{2} + \frac{x^6}{3} + \cdots\right) - x^3}{\left(\frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots\right)\left(-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} \\
&= \lim_{x\to 0} \frac{-\frac{5}{6}x^5 + \text{higher degree terms in } x}{-\frac{1}{12}x^5 + \text{higher degree terms in } x} \\
&= 10.
\end{aligned}$$