Q-1) Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^n n!} x^n$.

(i) Find its radius of convergence. (*4 points.*)

(ii) Find its interval of convergence. (*16 points.*)

You may use the following facts if you need them.

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi}\, n^{(n+1/2)}\, e^{-n}} = 1. \quad \text{(Stirling's Formula)}$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e, \text{ and moreover } \left(1+\frac{1}{n}\right)^n < e \text{ for al } n \geq 1.$$

Solution:

Let $a_n(x) = \dfrac{n^n}{2^n n!} x^n$. Using the Ratio Test we find

$$\lim_{n\to\infty} \frac{|a_{n+1}(x)|}{|a_n(x)|} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \frac{|x|}{2} = \frac{e}{2}\,|x|.$$

Therefore the series converges when $\dfrac{e}{2}\,|x| < 1$, or when $|x| < \dfrac{2}{e}$. This gives the radius of convergence as $\dfrac{2}{e}$.

Letting $x = 2/e$, we get $a_n(2/e) = \dfrac{n^n}{e^n\, n!}$. Using Stirling's formula we see that by taking $\epsilon = 1/2$, there exists an index N such that for all $n \geq N$, we have

$$\frac{1}{2} < \frac{n^n}{e^n\, n!} (\sqrt{2\pi}\, n^{1/2}) < \frac{3}{2},$$

or equivalently

$$\frac{1}{2\sqrt{2\pi}} \frac{1}{n^{1/2}} < a_n(2/e) < \frac{3}{2\sqrt{2\pi}} \frac{1}{n^{1/2}}.$$

By the Comparison Test, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges (by p-test), $\displaystyle\sum_{n=1}^{\infty} a_n(2/e)$ also diverges.

When $x = -2/e$, we have $a_n(-2/e) = (-1)^n a_n(2/e)$. From the above inequality we see that $\lim_{n\to\infty} a_n(2/e) = 0$. Moreover, we see that

$$\frac{a_{n+1}(2/e)}{a_n(2/e)} = (1+\frac{1}{n})^n \frac{1}{e} < 1,$$

where the inequality follows from the information given on the cover page. This shows that $a_n(2/e)$ decreases as n increases. These together imply that by the Alternating Series Test, $\displaystyle\sum_{n=1}^{\infty} a_n(-2/e)$ converges.

Hence the interval of convergence is $[-2/e, 2/e)$.