Q-2 Let $f(x,y)$ be a differentiable function satisfying the conditions

$$f(m,n) = mn,\ f_x(m,n) = m-n,\ f_y(m,n) = m+2n$$

for all integers m and n. Moreover let $x(t)$ and $y(t)$ be differentiable functions satisfying

$$x(n) = n^2,\ x'(n) = 3n,\ y(n) = n^3,\ y'(n) = n^4$$

for all integers n. Note that the expressions for these functions at non-integer values will be totally different.

Let $\phi(t) = f(x(t), y(t))$ for $t \in \mathbb{R}$. Calculate $\phi'(n)$ where n is an integer.

Solution:

$$\begin{aligned}
\phi'(n) &= f_x(x(n), y(n))\, x'(n) + f_y(x(n), y(n))\, y'(n) \\
&= f_x(n^2, n^3)\,(3n) + f_y(n^2, n^3)\,(n^4) \\
&= (n^2 - n^3)(3n) + (n^2 + 2n^3)(n^4) \\
&= n^3(3 - 3n + n^3 + 2n^4).
\end{aligned}$$

In case you wonder about the existence of such functions, here is a sample:

$$f(x,y) = xy + \left(\frac{\sin 2\pi x}{2\pi}\right)(x - 2y) + \left(\frac{\sin 2\pi y}{2\pi}\right)(2y),$$

$$x(t) = t^2 + \frac{t}{2\pi}\sin 2\pi t,$$

$$y(t) = t^3 - \frac{3t^2}{2\pi}\sin 2\pi t + \frac{t^4}{2\pi}\sin 2\pi t.$$