**Q-3)** Calculate the limit when it exists:

(i) $$\lim_{(x,y)\to(0,0)} \frac{x\sin^2 y}{x^4+y^4}. \quad \textit{(10 points.)}$$

(ii) $$\lim_{(x,y)\to(0,0)} \frac{x\sin^5 y}{x^4+y^4}. \quad \textit{(10 points.)}$$

Note that you can only use the methods described in the textbook! ☹

**Solution:** First note that

$$\lim_{(x,y)\to(0,0)} \frac{x\sin^k y}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{xy^k}{x^4+y^4} \lim_{(x,y)\to(0,0)} \frac{\sin^k y}{y^k} = \lim_{(x,y)\to(0,0)} \frac{xy^k}{x^4+y^4}.$$

Therefore it suffices to examine this last limit only.

Let $k \geq 4$. In this case we have

$$0 \leq \left|\frac{xy^k}{x^4+y^4}\right| \leq \frac{y^4}{x^4+y^4}\left|xy^{k-4}\right| \leq \left|xy^{k-4}\right| \to 0 \quad \text{as} \quad (x,y)\to(0,0).$$

Hence

$$\lim_{(x,y)\to(0,0)} \frac{x\sin^k y}{x^4+y^4} = 0 \text{ by the Sandwich Theorem, when } k \geq 4.$$

Now let $0 \leq k < 4$. Consider the path $(x,y) = (t^{4-k}, \lambda t)$, where $\lambda$ is a fixed constant and $t$ is a free variable.

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)=(t^{4-k},\lambda t)}} \frac{x\sin^k y}{x^4+y^4} = \lim_{t\to0} \frac{\lambda^k t^4}{(t^{4-k})^4+\lambda^4 t^4} = \lim_{t\to0} \frac{\lambda^k}{(t^{12-4k})+\lambda^4} = \begin{cases} \frac{\lambda^k}{1+\lambda^4} & \text{if } k=3, \\ \lambda^{k-4} & \text{if } k=0,1,2. \end{cases}$$

Hence the limit depends on path of approach. Thus the limit does not exist.

Conclusion: The limit exists if and only if $k \geq 4$, and in that case the limit is 0.