Q-4) Let

$$P(x,y) = 6x^2y^5 - y\cos x\sqrt{1+\cos^2 x} + 2,\ Q(x,y) = 10x^3y^4 + \sin y\sqrt{1+\sin^2 y} - 11.$$

Evaluate the line integral

$$\int_C P(x,y)dx + Q(x,y)dy,$$

where C is the curve $C_1 + C_2 + C_3$ with
C_1 is the line segment from $(0,0)$ to $(\pi/2,0)$,
C_2 is the line segment from $(\pi/2,0)$ to $(\pi/2,1)$, and
C_3 is the path along the curve $x = \arcsin y$ from $(\pi/2,1)$ to $(0,0)$.

Solution:

Let $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le \sin x, 0 \le x \le \pi/2\ \}$. Then $C = \partial D$ and we have

$$\begin{aligned}
\int_C Pdx + Qdy &= \int\int_D (Q_x - P_y)dA \\
&= \int_0^{\pi/2}\int_0^{\sin x} \cos x\sqrt{1+\cos^2 x}\, dydx \\
&= \int_0^{\pi/2} \sin x\cos x\sqrt{1+\cos^2 x}\, dx \\
&= -\frac{1}{3}(1+\cos^2 x)^{3/2}\Big|_0^{\pi/2} \\
&= \frac{2\sqrt{2}-1}{3}.
\end{aligned}$$