Q-5) Let S be the part of the paraboloid $z = x^2+y^2$ that lies inside the cylinder $x^2+y^2 = 25$. Find the surface area of S.

Solution:

We parametrize this surface as $\vec{\mathbf{r}}(x, y) = (x, y, x^2 + y^2)$ where $x^2 + y^2 \leq 25$.

$\vec{\mathbf{r}}_x = (1, 0, 2x)$, $\vec{\mathbf{r}}_y = (0, 1, 2y)$.

$$\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} = (-2x, -2y, 1).$$

$|\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y| = \sqrt{1 + 4(x^2 + y^2)}$.

$$\begin{aligned}
\text{Surface Area} &= \int\int_{x^2+y^2\leq 25} |\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y| \, dA \\
&= \int\int_{x^2+y^2\leq 25} \sqrt{1 + 4(x^2 + y^2)} \, dA \\
&= \int_0^{2\pi} \int_0^5 \sqrt{1 + 4r^2} \, r \, dr d\theta \\
&= (2\pi) \left(\frac{(1 + 4r^2)^{3/2}}{12} \Big|_0^5 \right) \\
&= (2\pi)(\frac{101\sqrt{101} - 1}{12}) \\
&= \frac{101\sqrt{101} - 1}{6} \, \pi \\
&\approx 265.
\end{aligned}$$