Date: 23 July 2016, Saturday
Time: 12:30-14:30

NAME:...

STUDENT NO:.......................................

YOUR DEPARTMENT:..................................

Math 102 Calculus II – Final Exam – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Please Read Carefully:

(i) Check that there are **4** questions on your exam booklet.

(ii) Write your name on top of every page.

(iii) Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

(iv) Calculators and dictionaries are not allowed.

(v) Use both sides of the sheets on this booklet if necessary. No extra papers will be provided.

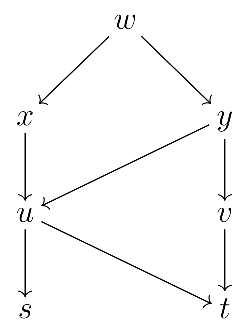
(vi) Turn off and leave your mobile phones with the exam proctor before the exam starts.

(vii) This exam is being video recorded. It is in your best interest not to give the slightest impression of doing anything improper, against the exam rules or the general rules of academic honesty.

(viii) **As you start the exam your grade is 0. You accumulate your grade as you write down meaningful answers. Contrary to common belief you do not start the exam with a grade of 100 and then the instructors *break* your grade down to what it is when you finally see your paper.**

NAME: STUDENT NO: DEPARTMENT:

Q-1) We have w as a function of x and y. Further x is a function of u, and y is a function of u and v. Finally u is a function of s and t, while v is a function of t. The dependence table of the variables is as given in the diagram here.



Use the table below to calculate the partial derivatives asked below the table. Simplify your answer and write only the resulting number into the boxes provided. Do not show your work. Only correct answers are graded in this questions, no partials!

$u(0,1)=3$	$u_s(0,1)=40$	$u_t(0,1)=-7$	$u(-1,2)=-3$	$u_s(-1,2)=5$	$u_t(-1,2)=-2$
$u(0,0)=6$	$u_s(0,0)=33$	$u_t(0,0)=44$	$v(1)=5$	$v_t(1)=-4$	$v(2)=0$
$v_t(2)=-19$	$v(0)=16$	$v_t(0)=8$	$x(3)=7$	$x_u(3)=11$	$x(-3)=11$
$x_u(-3)=4$	$x(6)=61$	$x_u(6)=3$	$y(3,5)=1$	$y_u(3,5)=13$	$y_v(3,5)=3$
$y(-3,0)=-1$	$y_u(-3,0)=30$	$y_v(-3,0)=3$	$y(6,16)=71$	$y_u(6,16)=-17$	$y_v(6,16)=0$
$w(7,1)=10$	$w_x(7,1)=1$	$w_y(7,1)=-1$	$w(11,-1)=12$	$w_x(11,-1)=3$	$w_y(11,-1)=2$
$w(61,71)=100$	$w_x(61,71)=14$	$w_y(61,71)=14$	$w(0,0)=23$	$w_x(0,0)=27$	$w_y(0,0)=29$

$$\left.\frac{\partial x}{\partial s}\right|_{(s,t)=(0,0)} = \boxed{99}$$ (6 points)

$$\left.\frac{\partial y}{\partial t}\right|_{(s,t)=(-1,2)} = \boxed{-117}$$ (6 points)

$$\left.\frac{\partial w}{\partial s}\right|_{(s,t)=(-1,2)} = \boxed{360}$$ (6 points)

$$\left.\frac{\partial w}{\partial t}\right|_{(s,t)=(0,1)} = \boxed{26}$$ (7 points)

NAME: STUDENT NO: DEPARTMENT:

Q-2) Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.

Solution:

First check that the point $p = (-1, 1, 2)$ is on both of the curves. Let C be the curve of intersection of this paraboloid and the ellipsoid. The tangent line L of C at p is orthogonal to the gradients of both the paraboloid and the ellipsoid, so it points in the direction of their cross-product.

Let $f = x^2 + y^2 - z$ and $g = 4x^2 + y^2 + z^2 - 9$.

$\nabla f = (2x, 2y, -1)$ and $\nabla g = (8x, 2y, 2z)$. (5+5 points)

$$\nabla f(p) \times \nabla g(p) = (-2, 2, -1) \times (-8, 2, 4) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & -1 \\ -8 & 2 & 4 \end{vmatrix} = (10, 16, 12). \qquad \text{(5+5 points)}$$

Thus we want to write the parametric equations of a line passing through the point $p = (-1, 1, 2)$ and pointing in the direction of $(10, 16, 12)$. We then get

$$L(t) = (-1, 1, 2) + t\,(10, 16, 12) = (-1 + 10t, 1 + 16t, 2 + 12t), \quad \text{where} \quad t \in \mathbb{R}. \qquad \text{(5 points)}$$

We can also write

$$\begin{aligned} x(t) &= -1 + 10t, \\ y(t) &= 1 + 16t, \\ z(t) &= 2 + 12t,\ t \in \mathbb{R}. \end{aligned}$$

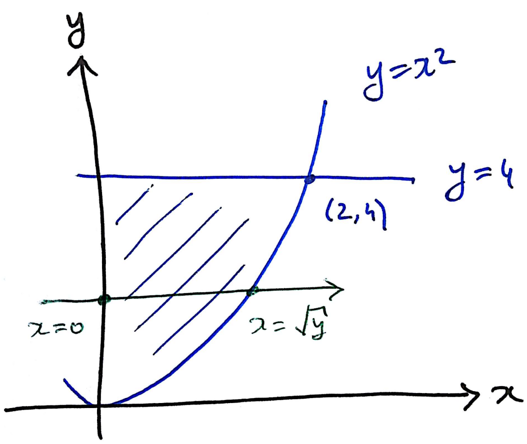
NAME: STUDENT NO: DEPARTMENT:

Q-3-a) Evaluate

$$\int_0^2 \int_{x^2}^4 x e^{y^2} \, dy dx. \qquad \text{(13 points)}$$

Q-3-a) Check for convergence $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{2016}}$. (12 points)

Solution-a:



$$\begin{aligned}
\int_0^2 \int_{x^2}^4 x e^{y^2} dy dx &= \int_0^4 \int_0^{\sqrt{y}} x e^{y^2} dx dy = \int_0^4 e^{y^2} \left(\frac{x^2}{2} \Big|_0^{\sqrt{y}} \right) dy \qquad \text{(3+3 points)} \\
&= \frac{1}{2} \int_0^4 e^{y^2} y \, dy = \frac{1}{4} \left(e^{y^2} \Big|_0^4 \right) \qquad \text{(2+2 points)} \\
&= \frac{1}{4}(e^{16} - 1) \qquad \text{(3 points)} \\
&\approx 2221527.380.
\end{aligned}$$

Solution-b:

By L'Hospital's rule we see that $\displaystyle\lim_{n\to\infty} \frac{(\ln n)^{2016}}{n} = 0$, so $n > (\ln n)^{2016}$ for all large n, or equivalently

$$\frac{1}{(\ln n)^{2016}} > \frac{1}{n} \quad \text{for all large } n. \qquad \text{(6 points)}$$

Thus our series diverges by direct comparison with the harmonic series. (6 points)

NAME: STUDENT NO: DEPARTMENT:

Q-4) Let S be the solid in the first octant bounded by the paraboloids $z = 9 - x^2 - y^2$, $z = 1 + x^2 + y^2$ and outside the cylinder $x^2 + y^2 = 1$. Evaluate

$$\int\int\int_S \frac{x}{x^2 + y^2}\, dV$$

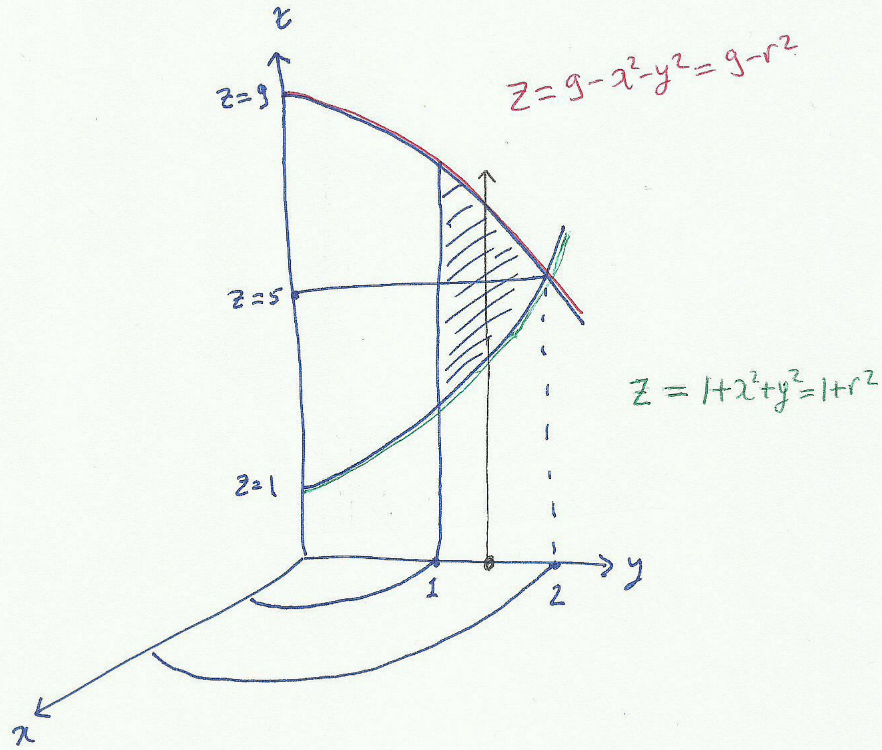
Solution:

We use cylindrical coordinates.

$$\begin{aligned}
\int\int\int_S \frac{x}{x^2 + y^2}\, dV &= \int_{\theta=0}^{\theta=\pi/2} \int_{r=1}^{r=2} \int_{z=1+r^2}^{r=9-r^2} \cos\theta \; dz dr d\theta \qquad \text{(10 points)} \\
&= \left(\sin\theta \Big|_0^{\pi/2} \right) \int_{r=1}^{r=2} (8 - 2r^2)\; dr \qquad \text{(5 points)} \\
&= \left(8r - \frac{2}{3} r^3 \Big|_1^2 \right) \qquad \text{(5 points)} \\
&= \frac{10}{3}. \qquad \text{(5 points)}
\end{aligned}$$

See the figure on next page.

Figure for Q4 in the final exam Math 102, summer 2016



S is the solid obtained by revolving the shaded region around the z-axis and taking the part that remains in the first octant.

A generic arrow parallel to the z-axis enters the region along the green surface $z = 1 + r^2$ and leaves the region along the red surface $z = 9 - r^2$.

The tip of this arrow now moves freely between the circles $r = 1$ and $r = 2$.