Date: 16 July 2016, Saturday
Time: 14:00-16:00

NAME:...

STUDENT NO:.......................................

YOUR DEPARTMENT:...................................

Math 102 Calculus II – Midterm Exam II – Solutions

1	2	3	4	TOTAL
25	25	25	25	100

Please do not write anything inside the above boxes!

Please Read Carefully:

(i) Check that there are **4** questions on your exam booklet.

(ii) Write your name on top of every page.

(iii) Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

(iv) Calculators and dictionaries are not allowed.

(v) Use both sides of the sheets on this booklet if necessary. No extra papers will be provided.

(vi) Turn off and leave your mobile phones with the exam proctor before the exam starts.

(vii) This exam is being video recorded. It is in your best interest not to give the slightest impression of doing anything improper, against the exam rules or the general rules of academic honesty.

(viii) **As you start the exam your grade is 0. You accumulate your grade as you write down meaningful answers. Contrary to common belief you do not start the exam with a grade of 100 and then the instructors *break* your grade down to what it is when you finally see your paper.**

NAME: STUDENT NO: DEPARTMENT:

Q-1) Write an equation for the plane which is perpendicular to both of the planes

$$x + 3y + 2z = 2016 \quad \text{and} \quad 2x - 3y + 2z = 2017,$$

and passing through the point $(1, 4, -2)$. Give your answer in the simplified form

$$Ax + By + Cz = D.$$

Grading: Finding the right criteria to find (A, B, C) is 10 points. Finding (A, B, C) correctly is 10 points. Finding D correctly is 5 points.

Solution:

The normal (A, B, C) of the required plane is perpendicular to both of the normals $(1, 3, 2)$ and $(2, -3, 2)$ of the given planes. (**10 points**)

Hence

$$(A, B, C) = (1, 3, 2) \times (2, -3, 2) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 3 & 2 \\ 2 & -3 & 2 \end{vmatrix} = (12, 2, -9). \quad \textbf{(10 points)}$$

The equation of the plane with this normal and passing through $(1, 4, -2)$ is then given by

$$12(x - 1) + 2(y - 4) - 9(z + 2) = 0,$$

which gives $D = 38$, (**5 points**). Finally the equation is

$$12x + 2y - 9z = 38.$$

NAME: STUDENT NO: DEPARTMENT:

Q-2-a) Find the value of x such that the vector $u = (1,1,x)$ makes an angle of $60°$ with the vector $v = (1,1,1)$. (12 points)

Q-2-b) Calculate the following limit, if it exists. $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin y}{x^2+y^4}$. (13 points)

Solution-a:

We want to solve the equation $u \cdot v = |u|\,|v|\,\cos 60°$. This gives

$$2 + x = \sqrt{2+x^2}\,\sqrt{3}\,\frac{1}{2}.$$

Simplifying this equation we get

$$x^2 + 16x + 10 = 0, \quad \textbf{(3 points)}$$

which gives the solutions

$$x_1 = -8 + \sqrt{54} \quad \text{and} \quad x_2 = -8 - \sqrt{54}. \quad \textbf{(3 points)}$$

Letting

$$u_1 = (1,1,x_1) \quad \text{and} \quad u_2 = (1,1,x_2)$$

we find that

$$u_1 \cdot v > 0 \quad \text{and} \quad u_2 \cdot v < 0. \quad \textbf{(3 points)}$$

This shows that u_1 makes an angle of $60°$ with v, whereas u_2 makes an angle of $120°$ with v. Hence the required answer is

$$x_1 = -8 + \sqrt{54} \approx -0.65. \quad \textbf{(3 points)}$$

Solution-b:

We use the squeeze theorem to show that the limit is zero. First note that

$$0 \le \frac{x^2}{x^2+y^4} \le 1 \quad \text{when} \quad (x,y) \ne (0,0).$$

Using this we have the inequalities

$$0 \le \left|\frac{x^2 \sin y}{x^2+y^4}\right| \le |\sin y|. \quad \textbf{(8 points)}$$

Now we have

$$\lim_{(x,y)\to(0,0)} |\sin y| = 0, \quad \textbf{(2 points)}$$

so by the squeeze theorem we get

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin y}{x^2+y^4} = 0. \quad \textbf{(3 points)}$$

NAME: STUDENT NO: DEPARTMENT:

Q-3) Let S be the surface in $\mathbb{R}^3$ given by the equation

$$5x^2 + 7y^2 - 9z^2 = 32.$$

(i) Write an equation for the tangent plane to the surface S at the point $(1, 2, -1/3) \in S$. Give your answer in the format $Ax + By + Cz = D$. **(5 points)**

(ii) Find the points on the surface S where the tangent plane is parallel to the plane $20x + 42y + 54z = 2016$. **(20 points)**

Solution:

(i) The gradient vector of the surface at the point (x, y, z) is given by

$$W(x, y, z) = (10x, 14y, -18z).$$

An equation for the tangent plane at the point $(1, 2, -1/3)$ is given by

$$W(1, 2, -1/3) \cdot (x - 1, y - 2, z + 1/3) = 0,$$

or

$$(10, 28, 6) \cdot (x, y, z) = (10, 28, 6) \cdot (1, 2, -1/3),$$

which finally gives the equation

$$10x + 28y + 6z = 64 \quad \text{or} \quad 5x + 14y + 3z = 32.$$

(ii) The gradient vector of the tangent plane at these points must be parallel to the normal vector of the given plane. This means that we are looking for the points $(x, y, z) \in S$ such that

$$W(x, y, z) = k(20, 42, 54) \quad \text{for some } k.$$

This gives

$$(10x, 14y, -18z) = (20k, 42k, 54k),$$

or we get

$$x = 2k, y = 3k, z = -3k.$$

Putting these into the equation of S we find $k^2 = 16$ or $k = \pm 4$. Hence the points on the surface S where the tangent plane is parallel to the given plane are

$$(8, 12, -12) \quad \text{and} \quad (-8, -12, 12).$$

NAME: STUDENT NO: DEPARTMENT:

Q-4) Consider the polynomial

$$f(x,y) = x^3 + 27y^3 + 108xy.$$

(i) Find all the critical points of f. (**8 points**)

(ii) Find the discriminant $D(x,y)$ of f (**9 points**)

(iii) Classify all the critical points. (**8 points**)

Solution:

(i) We want to solve the system

$$f_x = 3x^2 + 108y = 0 \quad \text{and} \quad f_y = 81y^2 + 108x = 0.$$

Solving for y from the first equation and substituting into the second equation we find

$$\left(\frac{x^3}{16} + 108\right) x = 0,$$

which give $x = 0$ or $x = -12$. Thus the critical points are

$$(x,y) = (0,0) \quad \text{and} \quad (x,y) = (-12,-4).$$

(ii) For the discriminant we first calculate

$$f_{xx} = 6x, f_{xy} = 108, f_{yy} = 162y.$$

Then the discriminant is

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 972\,xy - 108^2 = 972\,xy - 11664.$$

(iii) For the nature of the critical points we apply the second derivative test.

$$D(0,0) = -108^2 < 0, \quad \text{so } (0,0) \text{ is a saddle point,}$$

and

$$D(-12,-4) = 972(-12)(-4) - 108^2 = 34992 > 0, \quad \text{and} \quad f_{xx}(-12,-4) = -72 < 0,$$

so $(-12,-4)$ is a local maximum point.