Quiz # 6
Math 102-**002** Calculus
15 July 2016, Friday
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Solution Key

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Your Name: ....

Student ID: ....   Your Department: ....

Q-1) Let $f(x, y) = x^2 + 4xy + 5y^2 - 4x - 2y$.

(i) Find the critical points of f.

(ii) Determine the nature of the critical points.

(iii) Find the global minimum and maximum values of f if they exist.

Show your work in detail. Correct answers without justification are never graded.

Answer:

To find the critical points we must solve simultaneously the equations

$$f_x = 2x + 4y - 4 = 0 \quad \text{and} \quad f_y = 4x + 10y - 2 = 0.$$

Thus the only critical point is the sole solution to the above linear system which is $(x, y) = (8, -3)$.

To determine the nature of this critical point we need the second derivative test. For this we need

$$f_{xx} = 2,\ f_{xy} = 4,\ f_{yy} = 10,\ \Delta = f_{xx}f_{yy} - f_{xy}^2 = 4.$$

Since $\Delta > 0$, this critical point is a local minimum.

We can write $f(x, y) = (x + 2y)^2 + y^2 - 2(2x + y)$ and observe that as $|x|$ and $|y|$ increase, the values of the function go to $+\infty$. Hence there must be a global minimum which must occur at a critical point. Since we have only one critical point for this function, the global minimum whose existence we now know must occur there.

Hence $f(8, -3) = -13$ is the global minimum value of this function.

Note that the existence of a single local minimum point does not guarantee that there is a global minimum. Check for example that the polynomial $f(x, y) = x^2 + y^2(1 + x)^3$ has only one critical point at $(0, 0)$ which is a local minimum but is not a global minimum. In fact $\lim_{t\to\infty} f(-2, t) = -\infty$.