1. Consider the function

$$f(x, y, z) = x^3y^2z + ax^2y + bxz^2 \,,$$

where a and b are constants, the point $P_0(1, -1, 2)$, and the vector $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$.

a. Compute $\nabla f(P_0)$.

$$\vec{\nabla} f = (3x^2y^2z + 2axy + bz^2)\vec{i} + (2x^3yz + ax^2)\vec{j} + (x^3y^2 + 2bxz)\vec{k}$$

$$\vec{\nabla} f(P_0) = (6 - 2a + 4b)\vec{i} + (-4 + a)\vec{j} + (1 + 4b)\vec{k}$$

b. Find all possible values of (a, b) for which f increases the fastest in the direction of $\mathbf{A}$ at P_0.

We want $\vec{\nabla} f(P_0) = c \cdot \vec{A}$ with $c > 0$.

$$\vec{\nabla} f(P_0) \parallel \vec{A} \Rightarrow \frac{6 - 2a + 4b}{2} = \frac{-4 + a}{3} = \frac{1 + 4b}{6}$$

$$\Rightarrow \left\{ \begin{matrix} -8a + 12b = -26 \\ 6a - 12b = 27 \end{matrix} \right\} \Rightarrow a = -\frac{1}{2} \text{ and } b = -\frac{5}{2}$$

$$\Rightarrow \vec{\nabla} f(P_0) = -3\vec{i} - \frac{9}{2}\vec{j} - 9\vec{k} = -\frac{3}{2} \cdot \vec{A} \text{ and } c = -\frac{3}{2} < 0$$

$\Rightarrow$ There is no such (a, b)

c. Let $a = 3$ and $b = 1$. Find the directional derivative of f in the direction of $\mathbf{A}$ at P_0.

$$\vec{\nabla} f = 4\vec{i} - \vec{j} + 5\vec{k}$$

$$\vec{u} = (\text{the direction of } \vec{A}) = \frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{\sqrt{2^2 + 3^2 + 6^2}} = \frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{7}$$

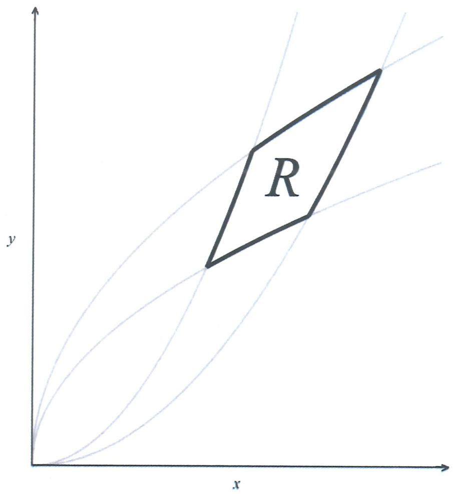
(The directional derivative of f at P_0 in the direction of $\vec{A}$)

$$= D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} = (4\vec{i} - \vec{j} + 5\vec{k}) \cdot \frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{7} = 5$$

2a. Evaluate the double integral

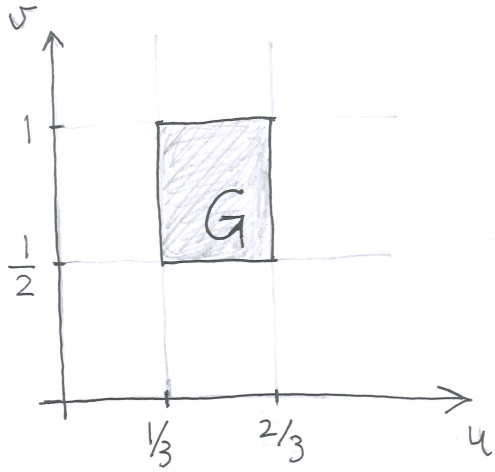
$$\iint_R \cos(\pi x^2/y)\, dA$$

where R is the region bounded by the parabolas $y = 3x^2$, $y = 3x^2/2$, $x = y^2$, $x = 2y^2$ in the plane.



$$\left.\begin{array}{l} u = \dfrac{x^2}{y} \\ v = \dfrac{y^2}{x} \end{array}\right\} \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 4-1=3$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{3}$$



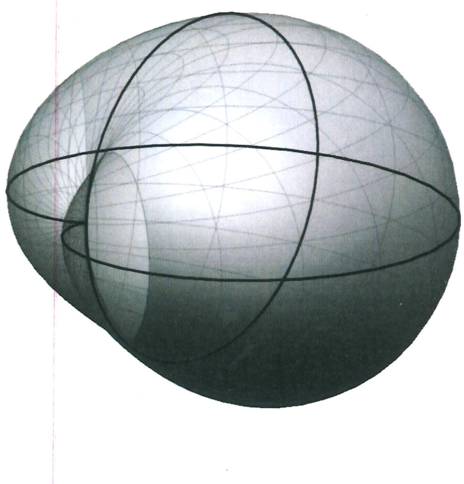
$$\iint_R \cos(\pi x^2/y)\, dA = \iint_G \cos(\pi u) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA$$

$$= \int_{1/2}^{1} \underbrace{\int_{1/3}^{2/3} \cos(\pi u) \cdot \frac{1}{3}\, du}_{\substack{\shortparallel \\ 0 \text{ by symmetry}}}\, dv = 0$$

2b. A solid D in space satisfies the following conditions:

- The intersection of D with the xy-plane is the region bounded by the cardioid with the equation $r = 1 + \cos\theta$ in polar coordinates.
- The intersection of D with each half-plane $\theta = c$ in spherical coordinates, where c is a constant, is a disk with a diameter lying in the xy-plane.

Express the volume V of the solid D as an iterated integral in spherical coordinates by filling in the rectangles below. No explanation is required.



$$V = \int_{\boxed{0}}^{\boxed{2\pi}} \int_{\boxed{0}}^{\boxed{\pi}} \int_{\boxed{0}}^{\boxed{(1+\cos\theta)\sin\phi}} \boxed{\rho^2 \sin\phi}\; d\rho\, d\phi\, d\theta$$

3. In each of the following, if the given statement is true for all sequences $\{a_n\}_{n=1}^{\infty}$, then mark the ☐ to the left of TRUE with a ✘; otherwise, mark the ☐ to the left of FALSE ✘ and give a counterexample. No explanation is required.

a. If $a_n < a_{n+1}$ for all $n \geq 1$, then $\lim_{n\to\infty} a_n = \infty$.

☐ TRUE

☒ FALSE, because it does not hold for $a_n =$ $\boxed{-\frac{1}{n}}$ for $n \geq 1$

b. If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

☐ TRUE

☒ FALSE, because it does not hold for $a_n =$ $\boxed{\frac{1}{n}}$ for $n \geq 1$

c. If $\sum_{n=1}^{\infty} a_n$ converges, then $\{a_n\}_{n=1}^{\infty}$ converges.

☒ TRUE

☐ FALSE, because it does not hold for $a_n =$ $\boxed{}$ for $n \geq 1$

d. If $0 < \frac{1}{2^n} < a_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

☐ TRUE

☒ FALSE, because it does not hold for $a_n =$ $\boxed{\frac{1}{2^{n-1}}}$ for $n \geq 1$

e. If $0 < a_n < \frac{1}{n}$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

☐ TRUE

☒ FALSE, because it does not hold for $a_n =$ $\boxed{\frac{1}{2n}}$ for $n \geq 1$

4. Determine whether each of the following series is convergent or divergent.

a. $\sum_{n=1}^{\infty} \cos\left(\frac{\pi}{n}\right)$

$$\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos 0 = 1 \neq 0 \Rightarrow \sum_{n=1}^{\infty} \cos\left(\frac{\pi}{n}\right) \text{ diverges by nTT.}$$

b. $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{\pi}{n}\right)\right)$

$$c = \lim_{n\to\infty} \frac{1-\cos\left(\frac{\pi}{n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2\sin^2\left(\frac{\pi}{2n}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\pi^2}{2}\cdot\left(\frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}}\right)^2 = \frac{\pi^2}{2}$$

$c = \frac{\pi^2}{2} < \infty$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=1}^{\infty}\left(1-\cos\left(\frac{\pi}{n}\right)\right)$ converges by LCT.

(p-series with $p = 2 > 1$)

c. $\sum_{n=1}^{\infty} \frac{1}{n^{2-\cos(\pi/n)}}$

$$c = \lim_{n\to\infty} \frac{\frac{1}{n^{2-\cos\left(\frac{\pi}{n}\right)}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{1-\cos\left(\frac{\pi}{n}\right)}} \overset{\circledast}{=} 1$$

$c = 1 > 0$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^{2-\cos\left(\frac{\pi}{n}\right)}}$ diverges by LCT.

(harmonic series)

$$\circledast \left\{ \begin{array}{l} \lim_{n\to\infty} \left(1-\cos\left(\frac{\pi}{n}\right)\right)\cdot \ln n = \lim_{n\to\infty} 2\sin^2\left(\frac{\pi}{2n}\right)\cdot \ln n = \lim_{n\to\infty} \frac{\pi^2}{2}\cdot\left(\frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}}\right)^2\cdot \underbrace{\frac{\ln n}{n^2}}_{\to\, 0} = 0 \\ \Rightarrow \lim_{n\to\infty} n^{1-\cos\frac{\pi}{n}} = e^0 = 1 \end{array} \right.$$

by "Useful Limits"

5. Consider the power series $f(x) = \sum_{n=2}^{\infty} \frac{x^n}{n^3 - n}$.

a. Find the interval of convergence I of the power series and determine whether it converges absolutely or conditionally at each point of I.

$$c_n = \frac{1}{n^3-n} \Rightarrow \frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^3-(n+1)}}{\frac{1}{n^3-n}} = \lim_{n\to\infty} \frac{n-1}{n+2} = 1 \Rightarrow R = 1$$

$\underline{x=1} \Rightarrow f(1) = \sum_{n=2}^{\infty} \frac{1}{n^3-n}$ converges by LCT because

$$c = \lim_{n\to\infty} \frac{\frac{1}{n^3-n}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1 < \infty$$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

(p-series with $p = 3 > 1$)

$\underline{x=-1} \Rightarrow f(-1) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^3-n}$ converges absolutely because

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n^3-n} \right| = \sum_{n=2}^{\infty} \frac{1}{n^3-n}$$ converges.

So: $f(x)$ converges absolutely at every point of its interval of convergence $I = [-1, 1]$.

b. Find the exact value of $f(-1)$.

$$f(-1) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^3-n} = \sum_{n=2}^{\infty} (-1)^n \left(\frac{n}{n^2-1} - \frac{1}{n} \right) = \sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{2} \cdot \frac{1}{n-1} + \frac{1}{2} \cdot \frac{1}{n+1} - \frac{1}{n} \right)$$

$$= \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1} + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^n}{n+1} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$= \frac{1}{2} \cdot \ln 2 + \frac{1}{2} \cdot \left(\ln 2 - \left(1 - \frac{1}{2}\right) \right) + \ln 2 - 1 = 2 \ln 2 - \frac{5}{4}$$

$$x = r\cos\theta \qquad r = \rho\sin\phi \qquad x = \rho\,\sin\phi\,\cos\theta$$
$$y = r\sin\theta \qquad \theta = \theta \qquad y = \rho\,\sin\phi\,\sin\theta$$
$$z = z \qquad z = \rho\cos\phi \qquad z = \rho\,\cos\phi$$

$$dA = dx\,dy = r\,dr\,d\theta$$
$$dV = dx\,dy\,dz = r\,dz\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$
$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}$$

$$\lim_{n\to\infty} x^n = 0 \text{ for all constant } |x| < 1$$
$$\lim_{n\to\infty} x^{1/n} = 1 \text{ for all constant } x > 0$$
$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for all constant } x$$
$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \text{ for all constant } x$$
$$\lim_{n\to\infty} \frac{(\ln n)^a}{n^b} = 0 \text{ for all constants } a \text{ and } b > 0$$
$$\lim_{n\to\infty} n^{1/n} = 1$$
$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$
$$\frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$$
$$\frac{1}{R} = \lim_{n\to\infty} |c_n|^{1/n}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{for } |x| < 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad \text{for } -1 < x \le 1$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } |x| \le 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \text{for all } x$$