1. Evaluate the following limits.

a. $\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^4+y^6} = 0$ by Sertöz Theorem as $\frac{1}{4}+\frac{5}{6}=\frac{13}{12}>1$

b. $\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^4+x^5y+y^6} = \lim_{(x,y)\to(0,0)} \left(\frac{xy^5}{x^4+y^6} \cdot \frac{x^4+y^6}{x^4+x^5y+y^6} \right)$

$= \underbrace{\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^4+y^6}}_{=0} \cdot \lim_{(x,y)\to(0,0)} \frac{1}{1+\frac{x^5y}{x^4+y^6}} = 0 \cdot 1 = 0$

$\frac{x^5y}{x^4+y^6} \to 0$

by Part a

by Sertöz Theorem as $\frac{5}{4}+\frac{1}{6}=\frac{17}{12}>1$

c. $\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^4+x^3y+y^6}$

$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^4+x^3y+y^6} = \lim_{x\to0} \frac{x\cdot0^5}{x^4+x^3\cdot0+0^6} = \lim_{x\to0} 0 = 0$

along the x-axis

$\lim_{(x,y)\to(0,0)} \frac{xy^5}{x^4+x^3y+y^6} = \lim_{x\to0} \frac{x\cdot(-x)^5}{x^4+x^3\cdot(-x)+(-x)^6} = \lim_{x\to0} -1 = -1$

along the line $y=-x$

$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy^5}{x^4+x^3y+y^6}$ does not exist by 2-Path Test

2. The *Tricomi equation*

$$yu_{xx} + u_{yy} = 0$$

arises in the study of transonic flow in fluid mechanics and in the study of isometric embeddings of 2-dimensional Riemannian manifolds into 3-dimensional Euclidian space in differential geometry.

Find all possible values of the pair of constants (a, b) for which the function $u(x, y) = (ax^2 + y^3)^b$ satisfies the Tricomi equation for all (x, y) with $ax^2 + y^3 > 0$.

$$u_x = b(ax^2+y^3)^{b-1} \cdot 2ax$$

$$u_{xx} = b\cdot(b-1)(ax^2+y^3)^{b-2}\cdot(2ax)^2 + b(ax^2+y^3)^{b-1}\cdot 2a$$

$$u_y = b(ax^2+y^3)^{b-1}\cdot 3y^2$$

$$u_{yy} = b\cdot(b-1)(ax^2+y^3)^{b-2}\cdot(3y^2)^2 + b(ax^2+y^3)^{b-1}\cdot 6y$$

$$yu_{xx}+u_{yy} = by(ax^2+y^3)^{b-2}\cdot\left((b-1)(4a^2x^2+9y^3)+(ax^2+y^3)\cdot(2a+6)\right)$$

$$= by(ax^2+y^3)^{b-2}\cdot\left(\{(b-1)\cdot 4a^2+2(a+3)\cdot a\}x^2+\{9(b-1)+2(a+3)\}y^3\right)$$

$yu_{xx}+u_{yy}=0$ for all (x,y) with $ax^2+y^3>0$

$\Leftrightarrow$ $b=0$ or $\left(4a^2(b-1)+2a(a+3)=0 \text{ and } 9(b-1)+2(a+3)=0\right)$

$a(4a-9)(b-1)=0 \Rightarrow a=0$ or $a=\frac{9}{4}$ or $b=1$

$a=0 \Downarrow b=\frac{1}{3}$; $a=\frac{9}{4} \Downarrow b=-\frac{1}{6}$; $b=1 \Downarrow a=-3$

u satisfies the Tricomi equation exactly when

$(a,b)=(0,\frac{1}{3}), (\frac{9}{4},-\frac{1}{6}), (-3,1), (a,0)$

a arbitrary

3. Consider the surfaces $S_1 : xyz = 10$ and $S_2 : z = x^2 + y^2$, and the point $P_0(1, 2, 5)$.

a. Find an equation of the tangent plane to S_1 at P_0.

$F(x,y,z) = xyz \Rightarrow \vec{\nabla} F = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$

$\Rightarrow \vec{n}_1 = \vec{\nabla} F(P_0) = 10\,\vec{i} + 5\,\vec{j} + 2\,\vec{k}$ is normal to S_1 at P_0 (1)

An equation of the tangent plane to S_1 at P_0 is:

$$10 \cdot (x-1) + 5 \cdot (y-2) + 2 \cdot (z-5) = 0$$

b. Find parametric equations of the tangent line to the curve of intersection of S_1 and S_2 at P_0.

$G(x,y,z) = x^2 + y^2 - z \Rightarrow \vec{\nabla} G = 2x\,\vec{i} + 2y\,\vec{j} - \vec{k}$

$\Rightarrow \vec{n}_2 = \vec{\nabla} G(P_0) = 2\,\vec{i} + 4\,\vec{j} - \vec{k}$ is normal to S_2 at P_0 (2)

(1) and (2) $\Rightarrow$ $\vec{v} = \vec{n}_1 \times \vec{n}_2$ is tangent to the curve of intersection of S_1 and S_2 at P_0.

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 10 & 5 & 2 \\ 2 & 4 & -1 \end{vmatrix} = -13\,\vec{i} + 14\,\vec{j} + 30\,\vec{k}$$

Parametric equations of the tangent line are:

$$\left.\begin{aligned} x &= -13t + 1 \\ y &= 14t + 2 \\ z &= 30t + 5 \end{aligned}\right\}$$

$-\infty < t < \infty$

4. Find the absolute maximum and minimum values of the function $f(x, y) = x^3 - y^2 + x^2y$ on the closed triangular region T shown in the figure below.

Interior of T:

$f_x = 3x^2 + 2xy = 0$

$f_y = -2y + x^2 = 0 \Rightarrow 2y = x^2$

$3x^2 + x^3 = 0 \Rightarrow x = 0$ or $x = -3$

$\Downarrow$ $y = 0$, $\Downarrow$ $y = \frac{9}{2}$

$\Downarrow$

$(x,y) = (0,0)$, $\cancel{(-3, \frac{9}{2})}$ not in T

Boundary of T:

Side 1: $-2 \le x \le 1$ and $y = 1$

$f(x,1) = x^3 - 1 + x^2$ for $-2 \le x \le 1$

$\frac{d}{dx} f(x,1) = 3x^2 + 2x = 0 \Rightarrow x = 0$ or $x = -\frac{2}{3}$

Endpoints: $x = -2$, $x = 1$

$\Rightarrow (x,y) = (0,1), (-\frac{2}{3}, 1), (-2, 1), (1,1)$

Side 2: $x = 1$ and $-2 \le y \le 1$

$f(1,y) = 1 - y^2 + y$ for $-2 \le y \le 1$

$\frac{d}{dy} f(1,y) = -2y + 1 = 0 \Rightarrow y = \frac{1}{2}$

Endpoints: $y = -2$, $y = 1$

$\Rightarrow (x,y) = (1, \frac{1}{2}), (1,-2), (1,1)$

Side 3: $y = -x - 1$ and $-2 \le x \le 1$

$f(x, -x-1) = -2x^2 - 2x - 1$ for $-2 \le x \le 1$

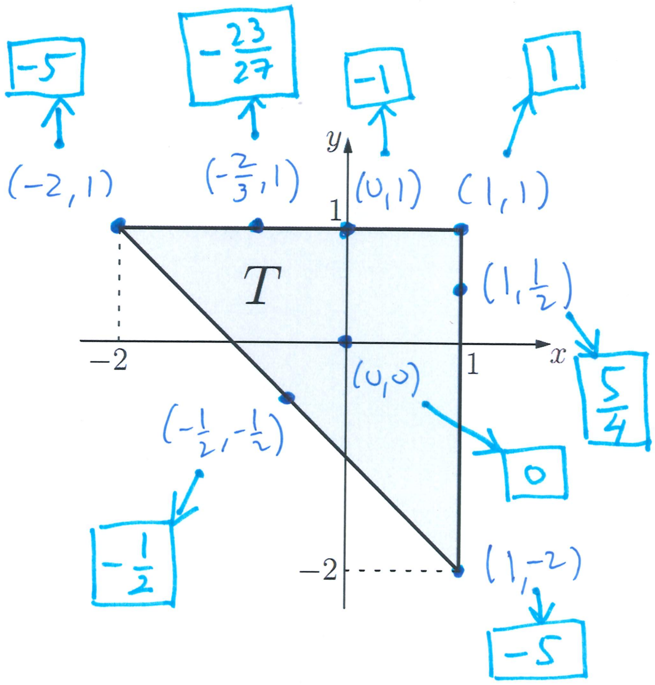
$\frac{d}{dx} f(x,-x-1) = -4x - 2 = 0 \Rightarrow x = -\frac{1}{2}$

Endpoints: $x = -2$, $x = 1$

$\Rightarrow (x,y) = (-\frac{1}{2}, -\frac{1}{2}), (-2,1), (1,-2)$

Abs. max is $\frac{5}{4}$

Abs. min is -5



Bonus. Estimate your total score for **Questions 1-4**. Your estimate must be an integer in the interval $[0, 100]$.

Write your estimate in the box ☞ $E = \boxed{}$

If your actual total score is T and

- if $|T - E| \leq 2$, then you will get *5 points*;
- if $2 < |T - E| \leq 5$, then you will get *2 points*

from this question.

Sertöz Theorem:

Let a and b be nonnegative integers, let c and d be positive even integers, and let

$$f(x,y) = \frac{x^a y^b}{x^c + y^d} \,.$$

Then:

- If $\dfrac{a}{c} + \dfrac{b}{d} > 1$, then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.
- If $\dfrac{a}{c} + \dfrac{b}{d} \leq 1$, then $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

The Second Derivative Test for Local Extreme Values:

Suppose that $f(x,y)$ has continuous second order partial derivatives on an open region containing (a,b) and that $f_x(a,b) = 0 = f_y(a,b)$. Let

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}.$$

Then:

- f has a local minimum at (a,b) if $\Delta(a,b) > 0$ and $f_{xx}(a,b) > 0$.
- f has a local maximum at (a,b) if $\Delta(a,b) > 0$ and $f_{xx}(a,b) < 0$.
- f has a saddle point at (a,b) if $\Delta(a,b) < 0$.