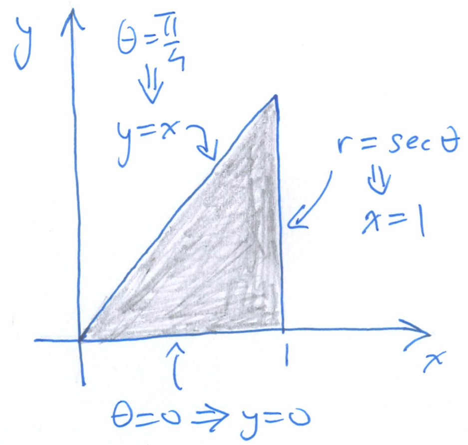
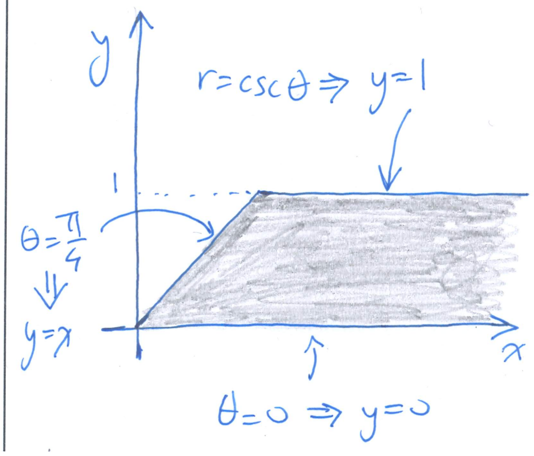
1. In each of the following, a double integral $\iint_D f(x, y)\, dA$ is expressed as an iterated integral in polar coordinates. In each part, draw a picture of the region D, and clearly label the curves bounding it with their equations both in Cartesian and polar coordinates.

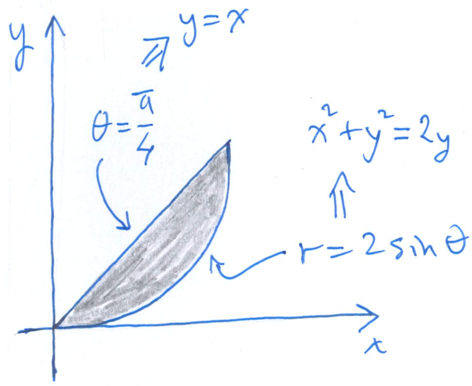
a. $\displaystyle\int_0^{\pi/4} \int_0^{\sec\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$



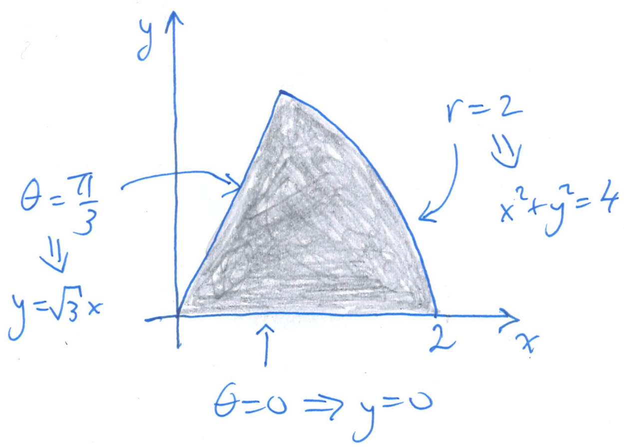
b. $\displaystyle\int_0^{\pi/4} \int_0^{\csc\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$



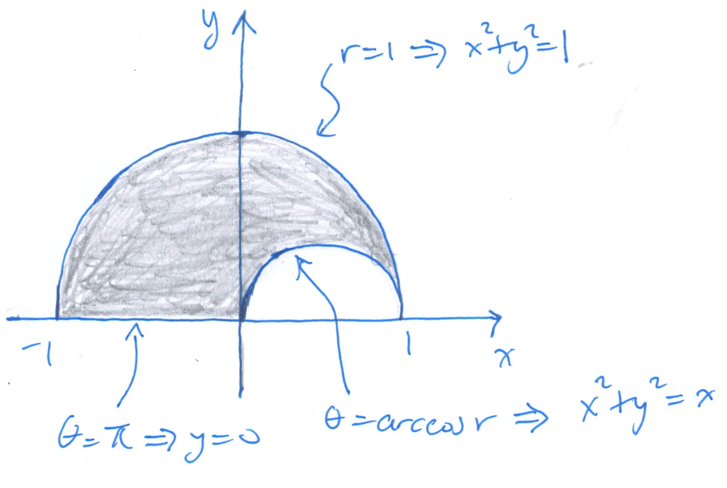
c. $\displaystyle\int_0^{\pi/4} \int_0^{2\sin\theta} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$



d. $\displaystyle\int_0^{\pi/3} \int_0^{2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$



e. $\displaystyle\int_0^{1} \int_{\arccos r}^{\pi} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$



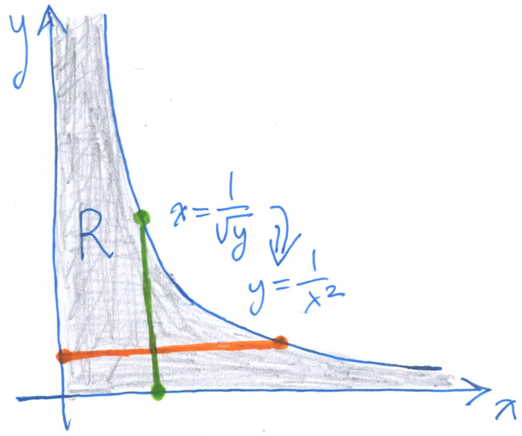
2a. Evaluate the iterated integral $\displaystyle\int_0^\infty \int_0^{1/\sqrt{y}} e^{-1/x}\,dx\,dy$. `[You can use the fact that` $\displaystyle\int_0^\infty e^{-x}\,dx = 1$.`]`

$$\int_0^\infty \int_0^{1/\sqrt{y}} e^{-1/x}\,dx\,dy = \iint_R e^{-1/x}\,dA = \int_0^\infty \int_0^{1/x^2} e^{-1/x}\,dy\,dx$$

$$= \int_0^\infty \left[e^{-1/x}y\right]_{y=0}^{y=1/x^2} dx = \int_0^\infty e^{-1/x}\cdot\frac{1}{x^2}\,dx$$

$$= \int_\infty^0 e^{-u}\cdot(-du) = 1$$

$$\boxed{\begin{aligned} u &= \frac{1}{x} \\ du &= -\frac{1}{x^2}\,dx \end{aligned}}$$



2b. Evaluate the double integral $\displaystyle\iint_R (x^2+y^2)\,dA$ where R is the region between the unit circle and the regular hexagon with center at the origin shown in the figure.

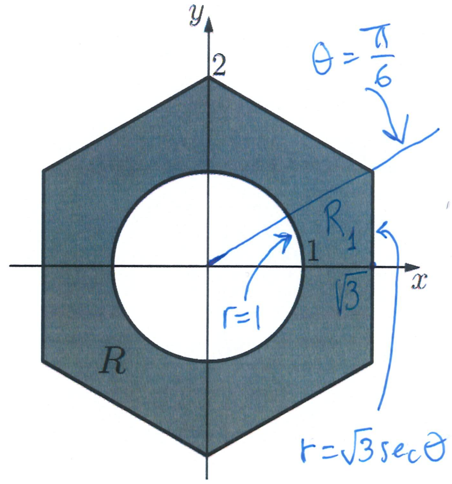
$$\iint_R (x^2+y^2)\,dA \overset{\text{by symmetry}}{=} 12\iint_{R_1}(x^2+y^2)\,dA = 12\int_0^{\pi/6}\int_0^{\sqrt{3}\sec\theta} r^2\cdot r\,dr\,d\theta$$

$$= 12\int_0^{\pi/6}\left[\frac{1}{4}r^4\right]_{r=0}^{r=\sqrt{3}\sec\theta} d\theta = 3\int_0^{\pi/6}(9\sec^4\theta - 1)\,d\theta$$

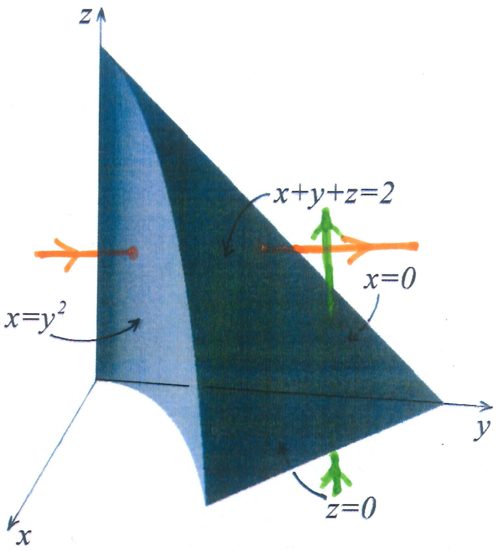
$$= 27\int_0^{\pi/6}(\tan^2\theta+1)\cdot\sec^2\theta\,d\theta - 3\int_0^{\pi/6} d\theta$$

$$= 27\left[\frac{1}{3}\tan^3\theta + \tan\theta\right]_0^{\pi/6} - 3\cdot\frac{\pi}{6}$$

$$= 27\cdot\left(\frac{1}{9\sqrt{3}}+\frac{1}{\sqrt{3}}\right) - \frac{\pi}{2} = 10\sqrt{3} - \frac{\pi}{2}$$



3. Let D be the region in space bounded by the parabolic cylinder $x = y^2$, the plane $x + y + z = 2$, the yz-plane, and the xy-plane.



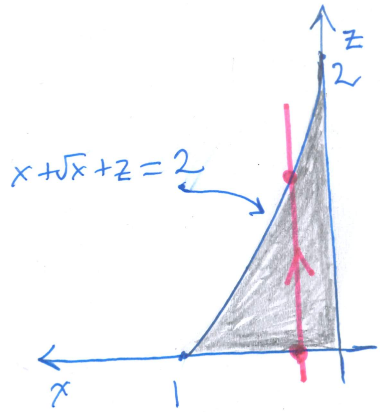
- Choose two of the following rectangular boxes by putting a ✗ in the ☐ in front of them, and then
- choose one of the orders of integration in each of the selected boxes by putting a ✗ in the ☐ in front of them.

☐	☐ $dx\,dy\,dz$ ☐ $dx\,dz\,dy$	☒	☐ $dy\,dx\,dz$ ☒ $dy\,dz\,dx$	☒	☐ $dz\,dx\,dy$ ☒ $dz\,dy\,dx$

Express the volume V of the region D in terms of iterated integrals in each of your selected orders of integration **(a)** and **(b)**.

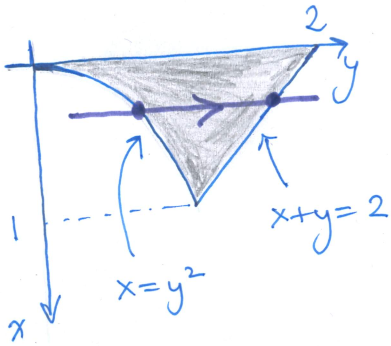
a.

$$V = \int_0^1 \int_0^{2-x-\sqrt{x}} \int_{\sqrt{x}}^{2-x-z} dy\, dz\, dx$$



b.

$$V = \int_0^1 \int_{\sqrt{x}}^{2-x} \int_0^{2-x-y} dz\, dy\, dx$$



c. Find the volume V.

$$V = \int_0^1 \int_0^{2-x-\sqrt{x}} (2-x-z-\sqrt{x})\, dz\, dx = \int_0^1 \Big[(2-x-\sqrt{x})z - \tfrac{1}{2}z^2\Big]_{z=0}^{z=2-x-\sqrt{x}} dx$$

$$= \int_0^1 \frac{1}{2}(2-x-\sqrt{x})^2\, dx = \int_0^1 \Big(2 + \frac{1}{2}x^2 + \frac{1}{2}x - 2x - 2\sqrt{x} + x^{3/2}\Big) dx$$

$$= 2 + \frac{1}{6} + \frac{1}{4} - 1 - \frac{4}{3} + \frac{2}{5} = \frac{29}{60}$$

4a. In ❶-❷, if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ satisfying the given conditions, write its n^{th} term in the box; and if no such sequence exists, write DOES NOT EXIST in the box. No explanation is required.

❶ $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 1$ and $\lim\limits_{n\to\infty} a_n$ does not exist.

$a_n = \boxed{n}$

❷ $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = -1$ and $\lim\limits_{n\to\infty} a_n$ exists.

$a_n = \boxed{\dfrac{(-1)^n}{n}}$

4b. Let c be a real number, and consider the sequence $\{a_n\}_{n=1}^{\infty}$ with $a_1 = c$ and satisfying the recursion relation $a_{n+1} = a_n + a_n^2$ for all $n \geq 1$.

① Show that if the sequence converges, then $\lim\limits_{n\to\infty} a_n = 0$.

Suppose $L = \lim\limits_{n\to\infty} a_n$. Then:

$a_{n+1} = a_n + a_n^2$ for all $n \geq 1 \Rightarrow L = L + L^2 \Rightarrow L^2 = 0 \Rightarrow L = 0$

② Fill in the boxes so that the sentence below becomes a true statement.

If $c = \boxed{\tfrac{1}{2}}$, *then the sequence* $\boxed{\text{diverges}}$.

Write **here** a real number which is not an integer

Write **here** either *converges* or *diverges*

③ Prove the statement in ②.

$a_1 = \frac{1}{2} \geq \frac{1}{2}$ and if $a_n \geq \frac{1}{2}$, then $a_{n+1} = a_n + a_n^2 \geq \frac{1}{2} + 0 = \frac{1}{2}$.

Hence, by induction, $a_n \geq \frac{1}{2}$ for all $n \geq 1$.

It follows that $\lim\limits_{n\to\infty} a_n \geq \frac{1}{2}$ if the limit exists,

Contradicting Part ①. Therefore the limit does not exist.

$$
\begin{aligned}
x &= r\cos\theta & r &= \rho\sin\phi & x &= \rho\,\sin\phi\,\cos\theta \\
y &= r\sin\theta & \theta &= \theta & y &= \rho\,\sin\phi\,\sin\theta \\
z &= z & z &= \rho\cos\phi & z &= \rho\,\cos\phi
\end{aligned}
$$

$$dA = dx\,dy = r\,dr\,d\theta$$

$$dV = dx\,dy\,dz = r\,dz\,dr\,d\theta = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$