Bilkent University

Quiz # 06
Math 102-Section **10** Calculus II
28 March 2019, Thursday
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Solution Key

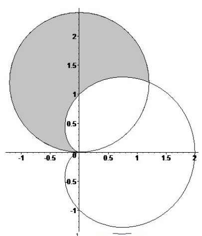
Q-1) Let $\alpha = \sqrt{2}+1$. The circle $r = \alpha\sin\theta$, $0 \le \theta \le \pi$, and the cardioid $r = 1+\cos\theta$, $0 \le \theta \le 2\pi$, intersect at two points one of which is $(0,\pi)$ in polar coordinates.

(i) Find the other intersection point.

(ii) Write a double integral which calculates the area inside the circle but outside the cardioid.

(iii) Find the exact value of this area by evaluating your double integral.

Grading: (i) 3 points, (ii) 5 points, (iii) 2 points.



Solution:

(i) By solving $\alpha\sin\theta = 1+\cos\theta$ we find that $\tan\dfrac{\theta}{2} = \dfrac{1}{\alpha}$. From the formula

$$\tan(a+b) = \frac{\tan a + \tan b}{1-\tan a\tan b},$$

by taking $a = b = \theta/2$, we find that $\tan\theta = 1$ and hence $\theta = \pi/4$ is the solution in the given range of θ.

(ii) Area of the shaded region is $A = \displaystyle\int_{\pi/4}^{\pi}\int_{1+\cos\theta}^{\alpha\sin\theta} r\,drd\theta.$

(iii)

$$\begin{aligned}
A &= \int_{\pi/4}^{\pi}\left(\frac{\alpha^2}{2}\sin^2\theta - \frac{1}{2}(1+\cos\theta)^2\right)d\theta \\
&= \int_{\pi/4}^{\pi}\left(\left(\frac{\alpha^2-3}{4}\right) - \left(\frac{\alpha^2+1}{4}\right)\cos 2\theta - \cos\theta\right)d\theta \\
&= \left(\left(\frac{\alpha^2-3}{4}\right)\theta - \left(\frac{\alpha^2+1}{8}\right)\sin 2\theta - \sin\theta\Big|_{\pi/4}^{\pi}\right) \\
&= \frac{\alpha^2-3}{4}\cdot\frac{3\pi}{4} + \frac{\alpha^2+1}{8} + \frac{1}{\sqrt{2}} \\
&= \frac{3\sqrt{2}\pi}{8} + \frac{1}{2} + \frac{3\sqrt{2}}{4} \approx 3.23.
\end{aligned}$$