Bilkent University

Quiz # 09
Math 102-Section **10** Calculus II
18 April 2019, Thursday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Define a sequence recursively by $a_1 = \sqrt{7}$, and $a_{n+1} = \sqrt{7a_n}$ for $n \geq 1$.

(i) Use induction to show that $0 < a_n < 7$ for all $n \geq 1$.

(ii) Use induction to show that (a_n) is a strictly increasing sequence.

(iii) Show that $\lim_{n\to\infty} a_n$ exists.

(iv) Calculate $\lim_{n\to\infty} a_n$.

Grading: (i) 3 points, (ii) 3 points, (iii) 3 points, (iv) 1 point.

Solution:

(i) It is clear that all $a_n > 0$. We prove that $a_n < 7$. For $n = 1$ we have $a_1 = \sqrt{7} < 7$. Assume $a_n < 7$ for some $n \geq 1$. We will check if $a_{n+1} < 7$ or not. We have $a_{n+1} = \sqrt{7a_n} < \sqrt{7 \cdot 7} = 7$ where we used $a_n < 7$. Hence by induction we proved that $a_n < 7$ for all $n \geq 1$.

(ii) First for $n = 1$ we have $a_2 = \sqrt{7a_1} > \sqrt{a_1 a_1} = a_1$ where we used $a_1 < 7$. So $a_2 > a_1$. Now assume $a_{n+1} > a_n$ for some $n \geq 1$. We will check if $a_{n+2} > a_{n+1}$ or not. We have $a_{n+2} - a_{n+1} = \sqrt{7a_{n+1}} - a_{n+1} > \sqrt{a_{n+1}a_{n+1}} - a_{n+1} = 0$ where we used $7 > a_{n+1}$ which we proved above. Hence we proved that a_n is strictly increasing

(iii) (a_n) is a bounded monotone sequence, so it converges.

(iv) Let $\lim_{n\to\infty} a_n = L$. Taking the limit of both sides of the recursive equation $a_{n+1} = \sqrt{7a_n}$ as $n \to \infty$ we get $L = \sqrt{7L}$, from which we conclude that $L = 7$ since $0 < a_1 < L$.