Bilkent University

Quiz # 04
Math 102 Section 03 Calculus II
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Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Let $F(x, y, z) = 7x^3 + 2xy - 4y^2 + 5xyz + 3z^6$. Assume that $F(x, y, z) + 12 = 0$ defines z as a differentiable function of x and y at the point $p_0 = (1, 2, -1)$.

(i) Write an equation for the tangent plane to the surface $F(x, y, z) + 12 = 0$ at the point $p_0 = (1, 2, -1)$.

(ii) Calculate $\left.\frac{\partial z}{\partial x}\right|_{p_0}$ and $\left.\frac{\partial z}{\partial y}\right|_{p_0}$.

(iii) Write a linearization of z at p_0.

(iv) How does your answer to (i) relate to your answer to (iii)?

Solution:

(i) First check that $F(1, 2, -1) + 12 = 0$, so the point p_0 is on the surface. An equation for the tangent plane at p_0 is

$$\nabla F(p_0) \cdot (p - p_0) = 0,$$

where $\nabla F = (F_x, F_y, F_z)$ is the gradient vector and $p = (x, y, z)$. We calculate the gradient vector as

$$\nabla F = (21x^2 + 2y + 5yz,\ \ 2x - 8y + 5xz,\ \ 5xy + 18z^5),\ \text{ and }\ \nabla F(p_0) = (15, -19, -8).$$

An equation for the tangent plane at p_0 then becomes

$$15x - 19y - 8z + 15 = 0. \qquad (*)$$

(ii) To calculate $\left.\frac{\partial z}{\partial x}\right|_{p_0}$ we either take the derivative of both sides of $F(x, y, z) + 12 = 0$ with respect to x, treating z as a function of x and y, and then substitute p_0 to solve for $\left.\frac{\partial z}{\partial x}\right|_{p_0}$, or we use the shorthand formula $z_x = -F_x/F_z$ and substitute p_0. Similarly for z_y at p_0. The required numbers are already calculated as entries of $\nabla F(p_0)$ above. Hence

$$\left.\frac{\partial z}{\partial x}\right|_{p_0} = \frac{15}{8}, \qquad \left.\frac{\partial z}{\partial y}\right|_{p_0} = -\frac{19}{8}.$$

(iii) The required linearization is of the form

$$z(x, y) = z(p_0) + z_x(p_0)(x - 1) + z_y(p_0)(y - 2),$$

which becomes after substitution

$$z = -1 + \frac{15}{8}(x - 1) - \frac{19}{8}(y - 2). \qquad (**)$$

(iv) If you simplify equation $(**)$, you will get exactly the equation $(*)$.