Bilkent University

Quiz # 06
Math 102 Section 03 Calculus II
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Solution Key

Q-1) Let S_R be the sphere given by $x^2+y^2+z^2=R^2$, where $R>0$. For any h with $0\leq h\leq 2R$, let V_h denote the volume of the region that lies inside the sphere S_R and between the planes $z=R$ and $z=R-h$. Calculate V_h.

Solution:

The plane $z=R$ touches the sphere S_R at its North pole. The plane $z=R-h$ cuts the sphere along a circle whose equation is $x^2+y^2+(R-h)^2=R^2$, or equivalently $x^2+y^2=a^2$ where $a=\sqrt{2Rh-h^2}$. Now we can calculate the volume using cylindrical coordinates.

Notice that an arbitrary arrow parallel to the z-axis enters the region when $z=R-h$, and leaves the region when $z=\sqrt{R^2-x^2-y^2}=\sqrt{R^2-r^2}$.

Moreover, the shadow of the region whose volume we calculate is given in the xy-plane, as above, $x^2+y^2=a^2$ where $a=\sqrt{2Rh-h^2}$.

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\begin{aligned}
V_h &= \int_0^{2\pi}\int_0^{\sqrt{2Rh-h^2}}\int_{R-h}^{\sqrt{R^2-r^2}} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{2Rh-h^2}} \left[r\sqrt{R^2-r^2}-(R-h)r\right]\,dr\,d\theta \\
&= \int_0^{2\pi}\left(-\frac{1}{3}(R^2-r^2)^{3/2}-(R-h)\frac{r^2}{2}\Bigg|_0^{\sqrt{2Rh-h^2}}\right)\,d\theta \\
&= \int_0^{2\pi}\left(\frac{Rh^2}{2}-\frac{h^3}{6}\right)\,d\theta \\
&= 2\pi\left(\frac{Rh^2}{2}-\frac{h^3}{6}\right) \\
&= \frac{\pi h^2}{3}(3R-h).
\end{aligned}
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