1. Let $a_1 = 1$ and $a_n = 1 + \dfrac{8}{1 + a_{n-1}}$ for $n \geq 2$, and let $b_n = \dfrac{2^n - (-1)^n}{3}$ for $n \geq 1$.

a. Fill in the following boxes.

$a_2 = \boxed{5}$ $\quad a_3 = \boxed{\frac{7}{3}}$ $\quad a_4 = \boxed{\frac{17}{5}}$ $\quad a_5 = \boxed{\frac{31}{11}}$

b. Fill in the following boxes.

$b_2 = \boxed{1}$ $\quad b_3 = \boxed{3}$ $\quad b_4 = \boxed{5}$ $\quad b_5 = \boxed{11}$

c. Guess an explicit formula for a_n.

$a_n = \boxed{3 \cdot \dfrac{2^n + (-1)^n}{2^n - (-1)^n}}$ for $n \geq 1$

d. Prove that your guess in **Part c** is correct.

Proof by induction:

$$\circledast \; n = 1 \Rightarrow 3 \cdot \frac{2^n + (-1)^n}{2^n - (-1)^n} = 3 \cdot \frac{2 + (-1)}{2 - (-1)} = 3 \cdot \frac{1}{3} = 1 = a_1$$

$\circledast$ Suppose $a_k = 3 \cdot \dfrac{2^k + (-1)^k}{2^k - (-1)^k}$ for some $k \geq 1$. Then:

$$1 + a_k = 1 + 3 \cdot \frac{2^k + (-1)^k}{2^k - (-1)^k} = \frac{4 \cdot 2^k + 2 \cdot (-1)^k}{2^k - (-1)^k} \quad \text{and}$$

$$a_{k+1} = 1 + \frac{8}{1 + a_k} = 1 + \frac{8 \cdot 2^k - 8 \cdot (-1)^k}{4 \cdot 2^k + 2 \cdot (-1)^k} = \frac{12 \cdot 2^k - 6 \cdot (-1)^k}{4 \cdot 2^k + 2 \cdot (-1)^k} = 3 \cdot \frac{2^{k+1} + (-1)^{k+1}}{2^{k+1} - (-1)^{k+1}}$$

Hence $a_n = 3 \cdot \dfrac{2^n + (-1)^n}{2^n - (-1)^n}$ for all $n \geq 1$

e. Find $\lim\limits_{n \to \infty} a_n$.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} 3 \cdot \frac{2^n + (-1)^n}{2^n - (-1)^n} = 3 \cdot \lim_{n \to \infty} \frac{1 + (-1/2)^n}{1 - (-1/2)^n} = 3 \cdot \frac{1 + 0}{1 - 0} = 3$$

2. Determine whether each of the following series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

$$s_n=\left(\frac{1}{\sqrt{1}}-\cancel{\frac{1}{\sqrt{2}}}\right)+\left(\cancel{\frac{1}{\sqrt{2}}}-\cancel{\frac{1}{\sqrt{3}}}\right)+\cdots+\left(\cancel{\frac{1}{\sqrt{n}}}-\frac{1}{\sqrt{n+1}}\right)=1-\frac{1}{\sqrt{n+1}} \quad \text{for } n\geq 1$$

$$\Rightarrow \lim_{n\to\infty} s_n=\lim_{n\to\infty}\left(1-\frac{1}{\sqrt{n+1}}\right)=1-0=1$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)=1 \Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right) \text{ converges.}$$

b. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\cdot\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)\right|=\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right) \text{ converges by } \underline{\text{Part a}}.$$

$$\Rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right) \text{ converges by ACT.}$$

3. Determine whether each of the following series converges or diverges.

a. $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)^2$

$$0<\frac{1}{\sqrt{n+1}}<\frac{1}{\sqrt{n}}\leq 1 \Rightarrow 0<\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}<1 \text{ for } n\geq 1$$

$$\left.\begin{array}{l}\Rightarrow 0<\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)^2<\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}} \text{ for } n\geq 1 \\ \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right) \text{ converges by } \underline{\text{Q2}} \text{ } \underline{\text{Part a}}\end{array}\right\}\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)^2 \text{ converges by DCT.}$$

b. $\sum_{n=1}^{\infty}\sqrt{\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}}$

$$\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}=\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\cdot\sqrt{n+1}}=\frac{1}{\sqrt{n}\cdot\sqrt{n+1}\cdot(\sqrt{n+1}+\sqrt{n})}$$

$$c=\lim_{n\to\infty}\frac{\sqrt{\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}}}{\frac{1}{n^{3/4}}}=\left(\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}\cdot\left(\sqrt{1+\frac{1}{n}}+1\right)}\right)^{1/2}=\left(\frac{1}{\sqrt{1+0}\cdot(\sqrt{1+0}+1)}\right)^{1/2}=\frac{1}{\sqrt{2}}$$

Since $c=\frac{1}{\sqrt{2}}>0$ and $\sum_{n=1}^{\infty}\frac{1}{n^{3/4}}$ diverges (p-series with $p=\frac{3}{4}\leq 1$),

$\sum_{n=1}^{\infty}\sqrt{\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}}$ diverges by LCT.

4a. Find all x satisfying the equation $x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \cdots \overset{\circledast}{=} -\dfrac{11}{111}$.

$$x - x^3 - x^4 + x^6 + x^7 - x^9 - x^{10} + \cdots = (x - x^3)\cdot(1 - x^3 + x^6 - x^9 + \cdots) = \frac{x - x^3}{1 + x^3} = \frac{x - x^2}{1 - x + x^2}$$

if $|x| < 1$, and the series diverges if $|x| \geq 1$.

If $|x| < 1$, then:

$$\circledast \iff \frac{x - x^2}{1 - x + x^2} = -\frac{11}{111} \iff 100x^2 - 100x - 11 = 0 \iff x = \frac{11}{10} \text{ or } x = -\frac{1}{10}$$

Hence $x = -\frac{1}{10}$ is the only solution.

4b. Show that $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n^2 + 4} < 1$.

$$\sum_{n=3}^{\infty} \frac{1}{n^2+4} < \int_2^{\infty} \frac{dx}{x^2+4} \overset{\boxed{\begin{array}{l} x = 2u \\ dx = 2du \end{array}}}{=} \frac{1}{2}\int_1^{\infty} \frac{du}{u^2+1} = \frac{1}{2}\lim_{c \to \infty} \int_1^c \frac{du}{u^2+1} = \frac{1}{2}\lim_{c\to\infty} \Big[\arctan u\Big]_1^c$$

$$= \frac{1}{2}\lim_{c\to\infty} (\arctan c - \arctan 1) = \frac{1}{2}\cdot\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+4} = \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \sum_{n=3}^{\infty} \frac{1}{n^2+4} < \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{\pi}{8} < \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{2}{5} = \frac{39}{40} < 1$$

5. Find the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{(n+2^n)(n+2^{-n})}$.

$$c_n = \frac{1}{(n+2^n)\cdot(n+2^{-n})} \Rightarrow \frac{1}{R} = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n\to\infty} \frac{\frac{1}{(n+1+2^{n+1})\cdot(n+1+2^{-(n+1)})}}{\frac{1}{(n+2^n)\cdot(n+2^{-n})}}$$

$$= \frac{1}{2}\lim_{n\to\infty} \frac{(n\cdot 2^{-n}+1)\cdot(1+n^{-1}\cdot 2^{-n})}{((n+1)\cdot 2^{-(n+1)}+1)\cdot(1+n^{-1}+n^{-1}\cdot 2^{-(n+1)})} = \frac{1}{2}\cdot\frac{(0+1)\cdot(1+0)}{(0+1)\cdot(1+0+0)} = \frac{1}{2} \Rightarrow R=2$$

$\underline{x=2}$: $$\sum_{n=0}^{\infty} \frac{x^n}{(n+2^n)\cdot(n+2^{-n})} = \sum_{n=0}^{\infty} \frac{2^n}{(n+2^n)\cdot(n+2^{-n})} = \sum_{n=0}^{\infty} \frac{1}{(n\cdot 2^{-n}+1)\cdot(n+2^{-n})}$$

$$c = \lim_{n\to\infty} \frac{\frac{1}{(n\cdot 2^{-n}+1)\cdot(n+2^{-n})}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{(n\cdot 2^{-n}+1)\cdot(1+n^{-1}\cdot 2^{-n})} = \frac{1}{(0+1)\cdot(1+0)} = 1$$

Since $c=1>0$ and $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (the harmonic series),

$\sum_{n=1}^{\infty} \frac{2^n}{(n+2^n)\cdot(n+2^{-n})}$ diverges by LCT.

$\underline{x=-2}$: $$\sum_{n=0}^{\infty} \frac{x^n}{(n+2^n)\cdot(n+2^{-n})} = \sum_{n=0}^{\infty} \frac{(-2)^n}{(n+2^n)\cdot(n+2^{-n})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n\cdot 2^{-n}+1)\cdot(n+2^{-n})}$$

Let $b_n = \frac{1}{(n\cdot 2^{-n}+1)\cdot(n+2^{-n})}$. Then $b_n>0$ for $n\geq 1$ and $\lim_{n\to\infty} b_n = 0$.

Moreover $b_n > b_{n+1}$ for sufficiently large n because:

$((n+1)\cdot 2^{-(n+1)}+1)\cdot(n+1+2^{-(n+1)}) > 1\cdot(n+1) = n+1 > n+(n^2+1+n\cdot 2^{-n})\cdot 2^{-n} = (n\cdot 2^{-n}+1)\cdot(n+2^{-n})$ for sufficiently large n.

as $\lim_{n\to\infty}(n^2+1+n\cdot 2^{-n})\cdot 2^{-n} = 0 \Rightarrow (n^2+1+n\cdot 2^{-n})\cdot 2^{-n} < 1$ for sufficiently large n.

Hence $\sum_{n=0}^{\infty} \frac{(-2)^n}{(n+2^n)\cdot(n+2^{-n})}$ converges by AST.

The interval of convergence of the power series is $[-2,2)$.

[In this problem you may use the fact that $\lim_{n\to\infty} n^k 2^{-n} = 0$ for all constants k.]