Bilkent University

Quiz # 03
Math 102 - Calculus II - Section 03
24 February 2022 Thursday
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Solution Key

Q-1) Consider the power series $\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2^n}$.

(a) Find the radius of convergence R of this power series.

(b) Check for convergence at both end points, i.e. when $|x| = R$.

Grading: 5+5 points

Solutions:

(a) Let

$$a_n = \begin{cases} 1 & \text{when } n = 2^k \text{ for some positive integer } k, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{n=0}^{\infty}(-1)^n x^{2^n} = \sum_{n=0}^{\infty}(-1)^{\epsilon_n} a_n x^n, \quad \text{where } \epsilon_n = \begin{cases} k & \text{if } n = 2^k, \\ 0 & \text{if } n \neq 2^k. \end{cases}$$

Finally note that

$$|a_n x^n| = \begin{cases} |x^n| & \text{if } n = 2^k, \\ 0 & \text{if } n \neq 2^k \end{cases} \leq |x^n|.$$

Then using direct comparison with the geometric series $\displaystyle\sum_{n=0}^{\infty}|x|^n$ which converges for $|x| < 1$, we conclude that our series converges absolutely for $|x| < 1$.

Hence the radius of convergence is $R = 1$.

(b) When $|x| = 1$, our series becomes $1 - 1 + 1 - 1 + \cdots$ and diverges since the general term does not converge to zero.

Alternate solution (*sketch*): Let $a_n = x^{2^n}$ Then using the Ratio Test we get

$$\left|\frac{a_{n+1}}{a_n}\right| = x^2 < 1$$

for convergence. At the end points the general term does not go to zero.