Bilkent University

Quiz # 09
Math 102 - Calculus II - Section 03
14 April 2022 Thursday
Instructor: Ali Sinan Sertöz
Solution Key

Q-1) Consider the function $f(x, y) = x^2 - xy + y^2 + 9x - 6y + 19$.

(a) Find all critical points of f.

(b) Determine the nature of each critical point as local min or local max or saddle point.

(c) Show that f is unbounded from above.

(d) Show that $f(x, y) \geq -11$ for all $x, y \in \mathbb{R}$.
(Hint: Substituting $u + v$ for x, and $u - v$ for y may be helpful!)

Grading: 2+2+3+3 points

Solutions:

(a) We solve the system

$$f_x = 2x - y + 9 = 0, \quad f_y = -x + 2y - 6 = 0,$$

to find that $(-4, 1)$ is the only critical point.

(b) To calculate the discriminant we first find the second derivatives as

$$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = -1.$$

Hence the discriminant is

$$\Delta = f_{xx} f_{yy} - f_{xy}^2 = 3.$$

Since $f_{xx} > 0$ and $f_{xy} > 0$, the critical point $(-4, 1)$ is a local minimum by the second derivative test.

(c) If we set $y = 0$ and send x to infinity, we get

$$\lim_{x \to \infty} f(x, 0) = \infty.$$

Hence f is unbounded from above.

(d) We have

$$f(u + v, u - v) = u^2 + 3v^2 + 3u + 15v + 19.$$

This shows that f cannot go to $-\infty$ along any path since the dominating terms are $u^2 + 3v^2$ are always positive and go to ∞ as $|u|$ and $|v|$ go to infinity.

On the other hand $f(-4, 1) = -2$ which is the local minimum which now we know must be the global minimum. Hence $f(x, y) \geq -2 > -11$.

Note also that we can write, after completing to squares

$$f(u+v,u-v) = \left(u+\frac{3}{2}\right)^2 + 3\left(v+\frac{5}{2}\right)^2 - 2,$$

which immediately shows that $f(x,y) \geq -2$.