1. Consider the function $f(x,y,z) = xy^2e^{Az}$, where A is a constant, and the point $P_0(2,-1,0)$.

a. Compute $\nabla f(P_0)$.

$\vec{\nabla} f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k} = y^2 e^{Az} \vec{i} + 2xy e^{Az} \vec{j} + Axy^2 e^{Az} \vec{k}$

$\vec{\nabla} f(P_0) = (-1)^2 \cdot e^0 \vec{i} + 2 \cdot 2 \cdot (-1) \cdot e^0 \vec{j} + A \cdot 2 \cdot (-1)^2 \cdot e^0 \vec{k} = \vec{i} - 4\vec{j} + 2A\vec{k}$

b. Find A if $D_{\mathbf{u}} f(P_0) = 1$ for $\mathbf{u} = \dfrac{\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{3}$.

$D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} = (\vec{i} - 4\vec{j} + 2A\vec{k}) \cdot \dfrac{\vec{i}+2\vec{j}+2\vec{k}}{3} = \dfrac{1\cdot 1 + (-4)\cdot 2 + 2A \cdot 2}{3} = \dfrac{4A-7}{3}$

$D_{\vec{u}} f(P_0) = 1 \Rightarrow \dfrac{4A-7}{3} = 1 \Rightarrow A = \dfrac{5}{2}$

c. Find a unit vector $\mathbf{u}$ such that $D_{\mathbf{u}} f(P_0) = 0$ for all values of the constant A.

If $\vec{u} = \dfrac{4\vec{i}+\vec{j}}{\sqrt{17}}$, then $D_{\vec{u}} f(P_0) = \vec{\nabla} f(P_0) \cdot \vec{u} = 0$ for all A.

d. Is there an A such that the maximum rate of change of f at P_0 is 5? If YES, find one; if NO, explain why not.

YES.

The maximum rate of change of f at P_0 is $|\vec{\nabla} f(P_0)| = \sqrt{1^2 + (-4)^2 + (2A)^2} = \sqrt{4A^2+17}$

If we take $A = \sqrt{2}$, then this is 5.

e. Is there an A such that the maximum rate of change of f at P_0 is 4? If YES, find one; if NO, explain why not.

No.

(The maximum rate) $= \sqrt{4A^2+17} \geq \sqrt{17} > 4$ for all choices of A.

2. Suppose that the graph of a differentiable function $z = f(x,y)$ contains the parametric curves

$$C_1: \ \mathbf{r}_1 = (t+1)\,\mathbf{i} + (2t+1)\,\mathbf{j} + (2t^3-1)\,\mathbf{k}\,, \quad (-\infty < t < \infty)\,,$$

and

$$C_2: \ \mathbf{r}_2 = (2-t)\,\mathbf{i} + (3-4t+t^2)\,\mathbf{j} + (1-t^2)\,\mathbf{k}\,, \quad (-\infty < t < \infty)\,.$$

a. Find an equation for the tangent plane to the graph of $z = f(x,y)$ at the point $(x,y,z) = (2,3,1)$.

$$\vec{v}_1 = \frac{d\vec{r}_1}{dt} = \vec{i} + 2\vec{j} + 6t^2\vec{k} \Rightarrow \vec{v}_1\Big|_{t=1} = \vec{i} + 2\vec{j} + 6\vec{k}$$

$$\vec{v}_2 = \frac{d\vec{r}_2}{dt} = -\vec{i} + (-4+2t)\vec{j} - 2t\vec{k} \Rightarrow \vec{v}_2\Big|_{t=0} = -\vec{i} - 4\vec{j}$$

$$\vec{n} = \vec{v}_1\Big|_{t=1} \times \vec{v}_2\Big|_{t=0} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 6 \\ -1 & -4 & 0 \end{vmatrix} = 24\vec{i} - 6\vec{j} - 2\vec{k}$$

An equation for the tangent plane is:

$$24\cdot(x-2) + (-6)\cdot(y-3) + (-2)\cdot(z-1) = 0$$

OR

$$z = 12x - 3y - 14$$

b. Compute $\left.\dfrac{d}{dt} f(6/t, 27/t^2)\right|_{t=3}$.

$$\Rightarrow f_x(2,3) = 12\,, \quad f_y(2,3) = -3$$

$$\frac{d}{dt} f(6/t, 27/t^2) = f_x(6/t, 27/t^2)\cdot\frac{d}{dt}(6/t) + f_y(6/t, 27/t^2)\cdot\frac{d}{dt}(27/t^2)$$

$$= f_x(6/t, 27/t^2)\cdot(-6/t^2) + f_y(6/t, 27/t^2)\cdot(-54/t^3)$$

$$\Rightarrow \frac{d}{dt} f(6/t, 27/t^2)\Big|_{t=3} = f_x(2,3)\cdot\left(-\frac{2}{3}\right) + f_y(2,3)\cdot(-2) = 12\cdot\left(-\frac{2}{3}\right) + (-3)\cdot(-2) = -2$$

3. Find and classify the critical points of $f(x, y) = 2xy^2 - 3x^2 - y^2 + x$.

$$\left.\begin{array}{l} (1)\ f_x = 2y^2 - 6x + 1 = 0 \\ (2)\ f_y = 4xy - 2y = 0 \end{array}\right\}$$

$(2) \Rightarrow 2 \cdot y \cdot (2x - 1) = 0 \Rightarrow y = 0$ or $x = \frac{1}{2}$

$y = 0$ and $(1) \Rightarrow -6x + 1 = 0 \Rightarrow x = \frac{1}{6} \Rightarrow (x, y) = (\frac{1}{6}, 0)$

$x = \frac{1}{2}$ and $(1) \Rightarrow 2y^2 - 2 = 0 \Rightarrow y^2 = 1 \Rightarrow y = 1$ or $y = -1$

$\Downarrow \qquad \Downarrow$

$(x, y) = (\frac{1}{2}, 1)$, $(\frac{1}{2}, -1)$

$$\Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -6 & 4y \\ 4y & 4x - 2 \end{vmatrix}$$

$\Delta(\frac{1}{6}, 0) = \begin{vmatrix} -6 & 0 \\ 0 & -\frac{4}{3} \end{vmatrix} = -6 \cdot (-\frac{4}{3}) - 0^2 = 8 > 0$ and $f_{xx}(\frac{1}{6}, 0) = -6 < 0$

$\Rightarrow f$ has a local max at $(\frac{1}{6}, 0)$.

$\Delta(\frac{1}{2}, 1) = \begin{vmatrix} -6 & 4 \\ 4 & 0 \end{vmatrix} = -6 \cdot 0 - 4^2 = -16 < 0 \Rightarrow f$ has a saddle point at $(\frac{1}{2}, 1)$.

$\Delta(\frac{1}{2}, -1) = \begin{vmatrix} -6 & -4 \\ -4 & 0 \end{vmatrix} = -6 \cdot 0 - (-4)^2 = -16 < 0 \Rightarrow f$ has a saddle point at $(\frac{1}{2}, -1)$.

$$\left.\begin{array}{lll} \Delta > 0 \text{ and } f_{xx} > 0 & \Longrightarrow & \text{local minimum} \\ \Delta > 0 \text{ and } f_{xx} < 0 & \Longrightarrow & \text{local maximum} \\ \Delta < 0 & \Longrightarrow & \text{saddle point} \end{array}\right\} \text{ where } \Delta = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

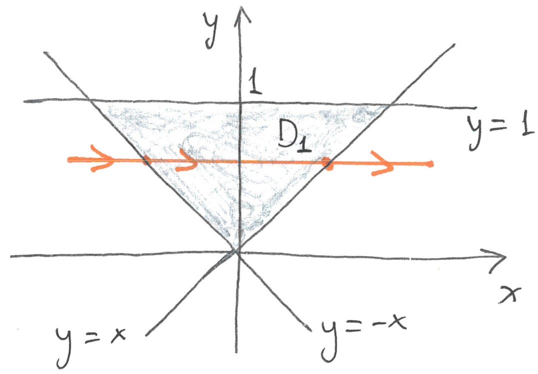
4a. Evaluate double integral $I_1 = \iint_{D_1} x^2 y \cos(\pi y^5/2)\, dA$ where $D_1 = \{(x,y) : -y \le x \le y \text{ and } y \le 1\}$.

$$I_1 = \int_0^1 \int_{-y}^{y} x^2 y \cos\left(\frac{\pi}{2} y^5\right) dx\, dy$$

$$= \int_0^1 \left[\frac{1}{3} x^3 y \cos\left(\frac{\pi}{2} y^5\right)\right]_{x=-y}^{x=y} dy$$

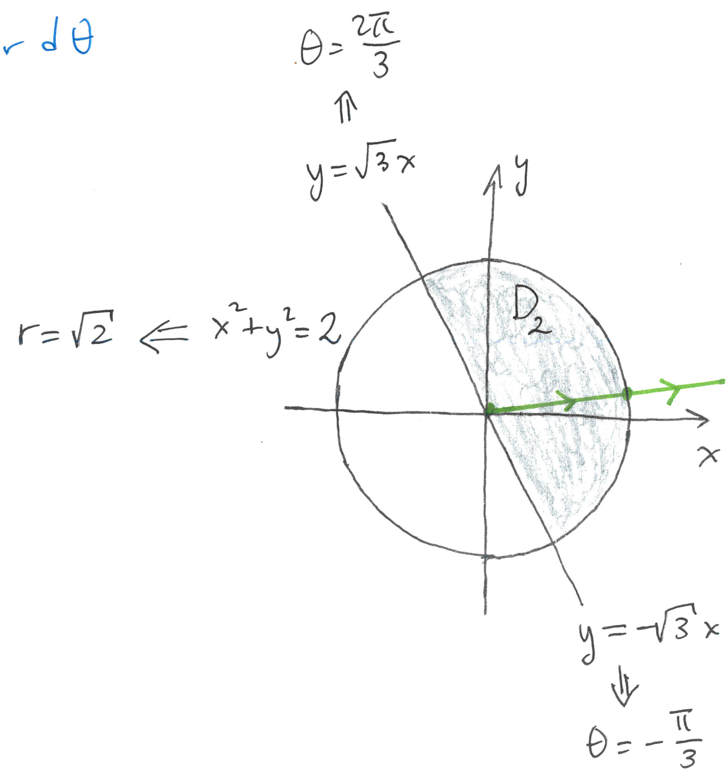
$$= \int_0^1 \left(\frac{1}{3} y^3 \cdot y \cos\left(\frac{\pi}{2} y^5\right) - \frac{1}{3} \cdot (-y)^3 y \cos\left(\frac{\pi}{2} y^5\right)\right) dy$$

$$= \int_0^1 \frac{2}{3} y^4 \cos\left(\frac{\pi}{2} y^5\right) dy = \frac{2}{3} \cdot \frac{2}{5\pi} \sin\left(\frac{\pi}{2} y^5\right)\Big]_0^1 = \frac{4}{15\pi} \sin\left(\frac{\pi}{2}\right) = \frac{4}{15\pi}$$



4b. Express the double integral $I_2 = \iint_{D_2} f(x,y)\, dA$ in terms of iterated integrals in polar coordinates where $D_2 = \{(x,y) : x^2 + y^2 \le 2 \text{ and } y + \sqrt{3}x \ge 0\}$.

$$I_2 = \int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} \int_0^{\sqrt{2}} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$



[6+6+8 *points*]

5. Let $E = \{(x, y, z) : x^2 + y^2 \le 3\,,\ 0 \le x \le y\,,\ \text{and}\ 0 \le z \le 3\}$.

a. Fill in the boxes so that the following equality holds for all continuous functions f where (x, y, z) are the rectangular coordinates.

$$\iiint_E f(x,y,z)\,dV = \int_{\boxed{0}}^{\boxed{\sqrt{3/2}}} \int_{\boxed{x}}^{\boxed{\sqrt{3-x^2}}} \int_{\boxed{0}}^{\boxed{3}} f(x,y,z)\,dz\,dy\,dx$$

b. Fill in the boxes so that the following equality holds for all continuous functions f where (r, θ, z) are the cylindrical coordinates.

$$\iiint_E f(x,y,z)\,dV = \int_{\boxed{\pi/4}}^{\boxed{\pi/2}} \int_{\boxed{0}}^{\boxed{\sqrt{3}}} \int_{\boxed{0}}^{\boxed{3}} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

c. Fill in the boxes so that the following equality holds for all continuous functions f where (ρ, ϕ, θ) are the spherical coordinates.

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_{\boxed{\pi/4}}^{\boxed{\pi/2}} \int_{\boxed{0}}^{\boxed{\pi/6}} \int_{\boxed{0}}^{\boxed{3/\cos\phi}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$+ \int_{\boxed{\pi/4}}^{\boxed{\pi/2}} \int_{\boxed{\pi/6}}^{\boxed{\pi/2}} \int_{\boxed{0}}^{\boxed{\sqrt{3}/\sin\phi}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$