1a. Determine whether the sequence $\left\{e^{1/n^2}\right\}_{n=1}^{\infty}$ converges or diverges.

$$\lim_{n\to\infty} e^{1/n^2} = e^{\lim_{n\to\infty}\frac{1}{n^2}} = e^0 = 1$$

$\left\{e^{1/n^2}\right\}_{n=1}^{\infty}$ converges.

1b. Determine whether the series $\sum_{n=1}^{\infty} e^{1/n^2}$ converges or diverges.

By Part a, $\lim_{n\to\infty} e^{1/n^2} = 1 \neq 0$.

Hence $\sum_{n=1}^{\infty} e^{1/n^2}$ diverges by n^{th} Term Test.

1c. Determine whether the series $\sum_{n=1}^{\infty}(e^{1/n^2} - 1)$ converges or diverges.

$$c = \lim_{n\to\infty} \frac{e^{1/n^2}-1}{\frac{1}{n^2}} = \lim_{x\to\infty} \frac{e^{1/x^2}-1}{\frac{1}{x^2}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^{1/x^2}\cdot(-2/x^3)}{(-2/x^3)} = \lim_{x\to\infty} e^{1/x^2} = 1$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges (p-series with $p=2>1$) and

$c = 1 < \infty$, $\sum_{n=1}^{\infty}(e^{1/n^2}-1)$ converges by Limit Comparison Test.

2. Determine whether each of the following series converges or diverges.

a. $\displaystyle\sum_{n=3}^{\infty} \frac{(n-2)^n(n+1)^n}{n^{2n+1}}$

useful limits

$$c = \lim_{n\to\infty} \frac{\frac{(n-2)^n(n+1)^n}{n^{2n+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \left(1-\frac{2}{n}\right)^n \cdot \left(1+\frac{1}{n}\right)^n = e^{-2} \cdot e = e^{-1}$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (the harmonic series) and

$c = e^{-1} > 0$, $\displaystyle\sum_{n=3}^{\infty} \frac{(n-2)^n(n+1)^n}{n^{2n+1}}$ diverges by Limit Comparison Test.

b. $\displaystyle\sum_{n=0}^{\infty} \frac{((2n)!)^2}{(3n)!\,n!}$

$$a_n = \frac{((2n)!)^2}{(3n)!\,n!}$$

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{((2\cdot(n+1))!)^2}{(3\cdot(n+1))!\,(n+1)!}\right|}{\left|\frac{((2n)!)^2}{(3n)!\,n!}\right|} = \lim_{n\to\infty} \left(\frac{(2n+2)!}{(2n)!}\right)^2 \cdot \frac{(3n)!}{(3n+3)!} \cdot \frac{n!}{(n+1)!}$$

$(2n+2)! = (2n+2)(2n+1)\cdot(2n)!$

$(3n+3)! = (3n+3)\cdot(3n+2)\cdot(3n+1)\cdot(3n)!$

$(n+1)! = (n+1)\cdot n!$

$$= \lim_{n\to\infty} \frac{((2n+2)(2n+1))^2}{(3n+3)(3n+2)(3n+1)(n+1)} = \frac{(2\cdot 2)^2}{3^3\cdot 1} = \frac{16}{27}$$

Since $L = \frac{16}{27} < 1$, $\displaystyle\sum_{n=0}^{\infty} \frac{((2n)!)^2}{(3n)!\,n!}$ converges by Ratio Test.

3. Determine the interval of convergence of the power series:

$$\sum_{n=0}^{\infty} \frac{2^n}{5^n + (-3)^n} x^n$$

$$a_n = \frac{2^n}{5^n + (-3)^n} x^n = \frac{1}{1 + \left(-\frac{3}{5}\right)^n} \cdot \frac{2^n x^n}{5^n}$$

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left|\frac{1}{1+\left(-\frac{3}{5}\right)^{n+1}} \cdot \frac{2^{n+1} x^{n+1}}{5^{n+1}}\right|}{\left|\frac{1}{1+\left(-\frac{3}{5}\right)^n} \cdot \frac{2^n x^n}{5^n}\right|} = \frac{2|x|}{5} \cdot \lim_{n\to\infty} \frac{1 + \overbrace{\left(-\frac{3}{5}\right)^n}^{\to 0}}{1 + \underbrace{\left(-\frac{3}{5}\right)^{n+1}}_{\to 0}} = \frac{2|x|}{5}$$

- If $|x| < \frac{5}{2}$, then $L < 1$ and the power series converges.
- If $|x| > \frac{5}{2}$, then $L > 1$ and the power series diverges.
- $x = \frac{5}{2} \Rightarrow \sum_{n=0}^{\infty} \frac{2^n}{5^n + (-3)^n} x^n = \sum_{n=0}^{\infty} \frac{2^n}{5^n + (-3)^n} \cdot \left(\frac{5}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{1 + \left(-\frac{3}{5}\right)^n}$

Since $\lim_{n\to\infty} \frac{1}{1+\left(-\frac{3}{5}\right)^n} = 1 \neq 0$, the power series diverges when $x = \frac{5}{2}$ by n^{th} Term Test.

- $x = -\frac{5}{2} \Rightarrow \sum_{n=0}^{\infty} \frac{2^n}{5^n + (-3)^n} x^n = \sum_{n=0}^{\infty} \frac{2^n}{5^n + (-3)^n} \cdot \left(-\frac{5}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{1 + \left(-\frac{3}{5}\right)^n}$

Since $\lim_{n\to\infty} \frac{1}{1+\left(-\frac{3}{5}\right)^n} = 1 \neq 0$, $\lim_{n\to\infty} \left((-1)^n \cdot \frac{1}{1+\left(-\frac{3}{5}\right)^n}\right)$ does not exist.

Hence the power series diverges when $x = -\frac{5}{2}$ by n^{th} Term Test.

The interval of convergence of the power series is $\left(-\frac{5}{2}, \frac{5}{2}\right)$.

4. Find an equation for the plane passing through the point $P_0(3,2,1)$ and containing the line of intersection of the planes $\mathscr{P}_1 : x+y+z=3$ and $\mathscr{P}_2 : x+2y+3z=6$.

$\uparrow$ (1) $\quad$ $\uparrow$ (2)

Consider the equation $a \times (1) + b \times (2)$ where a, b are constants, not both 0:

$$\mathscr{P}: \quad a\cdot(x+y+z) + b\cdot(x+2y+3z) = 3a+6b \qquad (3)$$

This equation defines a plane in space.

Since every point (x,y,z) which satisfies both equation (1) and equation (2) also satisfies equation (3), this plane contains the line of intersection of $\mathscr{P}_1$ and $\mathscr{P}_2$.

$\mathscr{P}$ contains the point P_0 exactly when $a\cdot(3+2+1)+b\cdot(3+2\cdot2+3\cdot1)=3a+6b$.

$$\Updownarrow$$

$$3a+4b=0$$

Let us choose $a=4$ and $b=-3$. Then

$$\mathscr{P}: \; x-2y-5z=-6$$

is an equation for the plane we seek.

5. The position vector of a point P as a function of time t is given by

$$\mathbf{r}(t) = \overrightarrow{OP}(t) = at\,\mathbf{i} + t^2\,\mathbf{j} + (t^3 + t)\,\mathbf{k} \quad (-\infty < t < \infty)$$

where a is a constant.

a. Compute the velocity vector $\mathbf{v}(t) = \dfrac{d}{dt}\mathbf{r}(t)$.

$\vec{v} = a\vec{i} + 2t\vec{j} + (3t^2+1)\vec{k}$

b. Suppose that the velocity vector $\mathbf{v}$ is parallel to the plane

$$\mathscr{P} : 3x + y + 4z = 1$$

at time $t = -2$. Find all other times t when the velocity vector $\mathbf{v}$ is parallel to this plane.

$\vec{n} = 3\vec{i} + \vec{j} + 4\vec{k}$ is a normal vector for $\mathscr{P}$.

$\vec{v} \parallel \mathscr{P} \iff \vec{v} \perp \vec{n} \iff \vec{v}\cdot\vec{n} = 0 \iff 3a + 1\cdot 2t + 4\cdot(3t^2+1) = 0$

$\vec{v}\big|_{t=-2} \parallel \mathscr{P} \Rightarrow 3a + 1\cdot 2\cdot(-2) + 4\cdot(3\cdot(-2)^2+1) = 0 \Rightarrow a = -16$

Hence $\vec{v} \parallel \mathscr{P} \iff 3\cdot(-16) + 1\cdot 2t + 4\cdot(3t^2+1) = 0 \iff 6t^2 + t - 22 = 0$

$\iff (t+2)(6t-11) = 0 \iff t = -2$ or $t = \dfrac{11}{6}$

$t = \dfrac{11}{6}$ is the only other time when $\vec{v} \parallel \mathscr{P}$

c. Suppose that there is a time t when the velocity vector $\mathbf{v}$ is parallel to the line

$$L : \frac{x-1}{5} = \frac{y-2}{-1} = \frac{z+1}{2}.$$

Find all values of the constant a that makes this possible.

$\vec{w} = 5\vec{i} - \vec{j} + 2\vec{k}$ is parallel to L.

$\vec{v} \parallel L \iff \vec{v} \parallel \vec{w} \iff \dfrac{a}{5} \underset{①}{=} \dfrac{2t}{-1} \underset{②}{=} \dfrac{3t^2+1}{2}$

② $\iff 3t^2 + 4t + 1 = 0 \iff (t+1)(3t+1) = 0 \iff t = -1$ or $t = -\dfrac{1}{3}$

In these cases, ① is also satisfied exactly when $a = 10$ and $a = \dfrac{10}{3}$, respectively.