1. Consider the function:

$$f(x,y) = 8x + 2xy - xy^2 - 3x^2$$

The complete list of the critical points and the values of f at these points is given in the table. [You do not need to verify these.]

(a,b)	$(0,-2)$	$(0,4)$	$(3/2,1)$
$f(a,b)$	0	0	$27/4$

Find the absolute maximum and minimum values of f on $D = \{(x,y) : -1 \le x \le 2 \text{ and } 0 \le y \le 3\}$.

Boundary of D:

Side 1: $y=0$ and $-1 \le x \le 2 \Rightarrow f(x,0) = 8x - 3x^2$ for $-1 \le x \le 2$

Critical points: $\frac{d}{dx} f(x,0) = 8 - 6x = 0 \Rightarrow x = \frac{4}{3} \Rightarrow (x,y) = (\frac{4}{3}, 0)$

Endpoints: $x=-1$, $x=2 \Rightarrow (x,y) = (-1,0), (2,0)$

Side 2: $y=3$ and $-1 \le x \le 2 \Rightarrow f(x,3) = 5x - 3x^2$ for $-1 \le x \le 2$

Critical points: $\frac{d}{dx} f(x,3) = 5 - 6x = 0 \Rightarrow x = \frac{5}{6} \Rightarrow (x,y) = (\frac{5}{6}, 3)$

Endpoints: $x=-1$, $x=2 \Rightarrow (x,y) = (-1,3), (2,3)$

Side 3: $x=-1$ and $0 \le y \le 3 \Rightarrow f(-1,y) = y^2 - 2y - 11$ for $0 \le y \le 3$

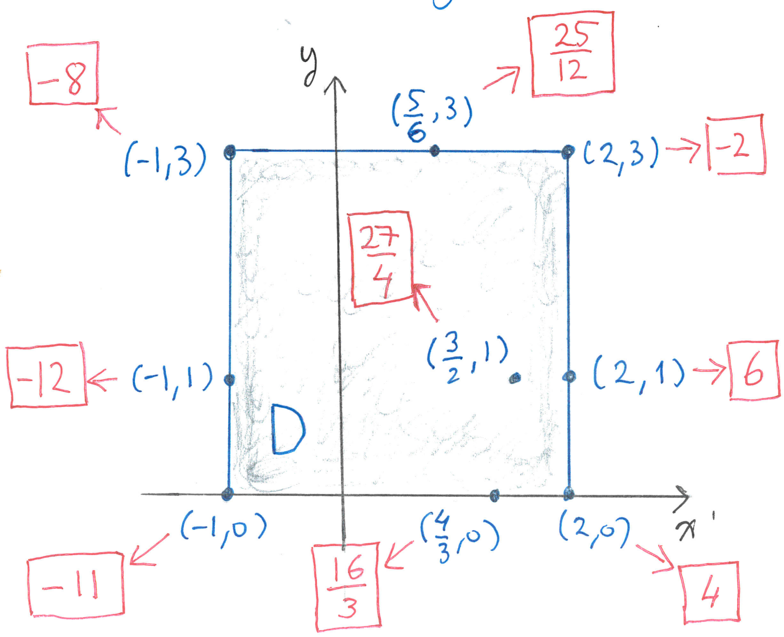
Critical points: $\frac{d}{dy} f(-1,y) = 2y - 2 = 0 \Rightarrow y = 1 \Rightarrow (x,y) = (-1,1)$

Endpoints: $y=0$, $y=3 \Rightarrow (x,y) = (-1,0), (-1,3)$

Side 4: $x=2$ and $0 \le y \le 3 \Rightarrow f(2,y) = -2y^2 + 4y + 4$ for $0 \le y \le 3$

Critical points: $\frac{d}{dy} f(2,y) = -4y + 4 = 0 \Rightarrow y = 1 \Rightarrow (x,y) = (2,1)$

Endpoints: $y=0$, $y=3 \Rightarrow (x,y) = (2,0), (2,3)$



Interior of D:

$(\frac{3}{2}, 1)$ is the only critical point in the interior of D

Absolute maximum is $\frac{27}{4}$.

Absolute minimum is -12.

2. Evaluate the following integrals.

a. $\iint_D x^3 e^{x^2 y}\, dA$ where $D = \{(x,y) : 0 \le x \le 2 \text{ and } 0 \le y \le 1/2\}$

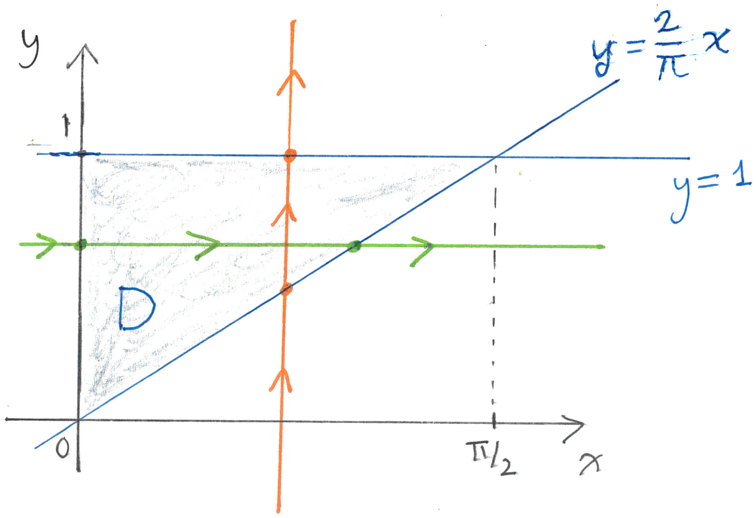
$$\iint_D x^3 e^{x^2 y}\, dA = \int_0^2 \int_0^{1/2} x^3 e^{x^2 y}\, dy\, dx = \int_0^2 x e^{x^2 y}\Big]_{y=0}^{y=1/2} dx$$

$$= \int_0^2 \left(x e^{x^2/2} - x\right) dx = \left[e^{x^2/2} - \frac{1}{2}x^2 \right]_0^2 = e^2 - 2 - (1-0) = e^2 - 3$$

b. $\int_0^{\pi/2} \int_{2x/\pi}^{1} \cos(x/y)\, dy\, dx = \iint_D \cos(x/y)\, dA = \int_0^1 \int_0^{\pi y/2} \cos(x/y)\, dx\, dy$

$$= \int_0^1 y \sin(x/y)\Big]_{x=0}^{x=\pi y/2} dy = \int_0^1 \left(y \sin(\pi/2) - y \sin(0)\right) dy = \int_0^1 y\, dy = \frac{1}{2}y^2\Big]_0^1$$

$$= \frac{1}{2} - 0 = \frac{1}{2}$$



3. Let $E = \{(x,y,z) : x^2+y^2+z^2 \le 2z \text{ and } x^2+y^2+z^2 \le 1 \text{ and } 0 \le x \le y\}$.

a. Fill in the boxes so that the following equality holds for all continuous functions f where (r,θ,z) are the cylindrical coordinates.

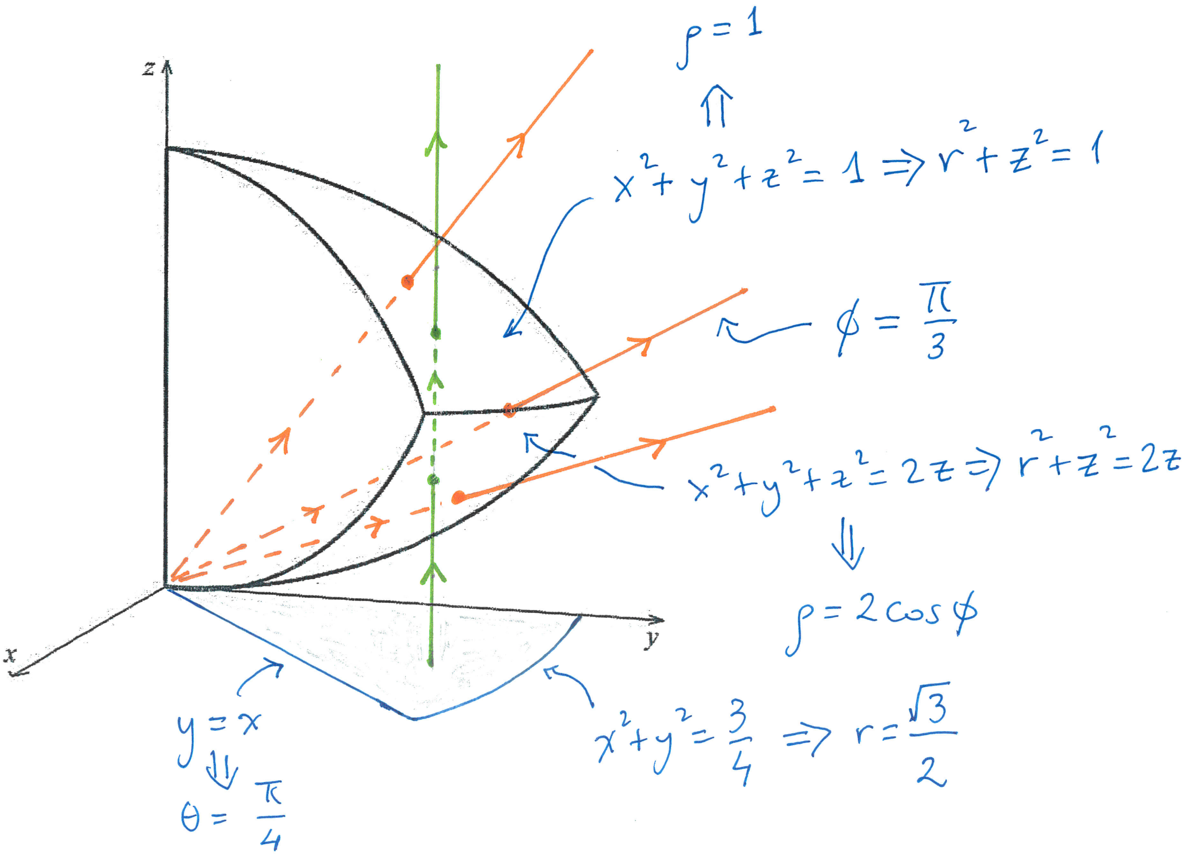
$$\iiint_E f(x,y,z)\,dV = \int_{\boxed{\pi/4}}^{\boxed{\pi/2}} \int_{\boxed{0}}^{\boxed{\sqrt{3}/2}} \int_{\boxed{1-\sqrt{1-r^2}}}^{\boxed{\sqrt{1-r^2}}} f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta$$

b. Fill in the boxes so that the following equality holds for all continuous functions f where (ρ,ϕ,θ) are the spherical coordinates.

$$\iiint_E f(x,y,z)\,dV$$

$$= \int_{\boxed{\pi/4}}^{\boxed{\pi/2}} \int_{\boxed{\pi/3}}^{\boxed{\pi/2}} \int_{\boxed{0}}^{\boxed{2\cos\phi}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$+ \int_{\boxed{\pi/4}}^{\boxed{\pi/2}} \int_{\boxed{0}}^{\boxed{\pi/3}} \int_{\boxed{0}}^{\boxed{1}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

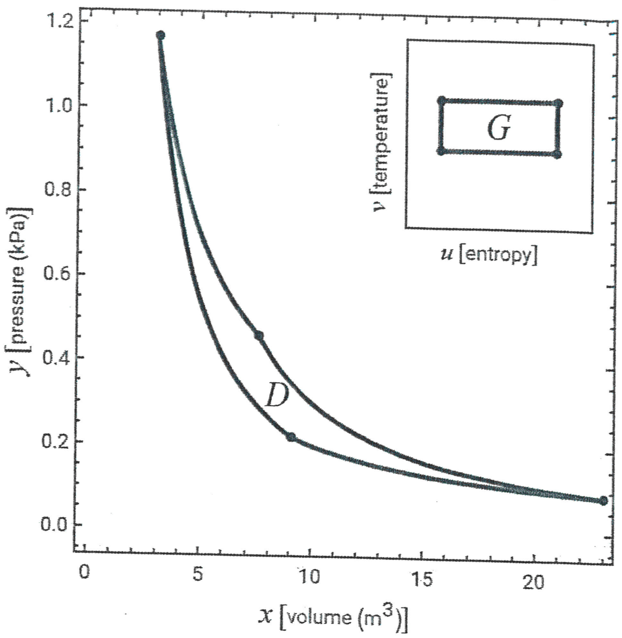


4. A *Carnot cycle* is a thermodynamic process consisting of consecutive isothermal and isentropic expansions and compressions modeling an ideal heat engine (or refrigerator).

The work done by the engine (or on the refrigerator) is equal to the area A of the region D enclosed by the curves

$$xy = 2\,,\ xy = 7/2\,,\ x^3y^2 = 12\,,\ \text{and}\ x^3y^2 = 30$$

in the xy-plane as shown in the figure.



Consider the transformation

$$u = x^3y^2 \ \text{and}\ v = xy$$

and let G be the region in the uv-plane corresponding to D under this transformation.

a. Compute $\dfrac{\partial(u,v)}{\partial(x,y)}$ in terms of x and y.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 3x^2y^2 & 2x^3y \\ y & x \end{vmatrix} = 3x^2y^2\cdot x - 2x^3y\cdot y = x^3y^2$$

b. Compute $\dfrac{\partial(x,y)}{\partial(u,v)}$ in terms of u and v.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{x^3y^2} = \frac{1}{u}$$

c. Compute A by expressing it as a double integral in the uv-plane over the region G.

$$A = \iint_D dx\,dy = \iint_G \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_2^{7/2}\int_{12}^{30} \left|\frac{1}{u}\right| du\,dv = \int_2^{7/2}\int_{12}^{30} \frac{1}{u}\,du\,dv$$

$$= \int_2^{7/2} \ln|u|\Big]_{u=12}^{u=30} dv = \int_2^{7/2} (\ln(30) - \ln(12))\,dv = \ln\left(\frac{5}{2}\right)\cdot v\Big]_2^{7/2}$$

$$= \ln\left(\frac{5}{2}\right)\cdot\left(\frac{7}{2} - 2\right) = \frac{3}{2}\ln\left(\frac{5}{2}\right)$$

$$\iint_D f(x,y)\,dx\,dy = \iint_G f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \quad\text{where}\quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$