Bilkent University

Quiz # 09
Math 102 Section 08 Calculus II
15 April 2024 Monday
Instructor: Ali Sinan Sertöz
**Solution Key**

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**Q-1)** Let $z = z(x, y)$, $x = x(u, v)$, $y = y(u, v)$, $u = u(s, t)$ and $v = v(s, t)$ be differentiable functions of their variables. We have the following data:

| | | | |
|---|---|---|---|
| $u(3,7) = 9$ | $v(3,7) = 11$ | $u(7,3) = -9$ | $u(7,3) = -10$ |
| $u_s(3,7) = 12$ | $v_s(3,7) = -1$ | $u_t(3,7) = 2$ | $v_t(3,7) = -3$ |
| $x(9,11) = -3$ | $y(9,11) = 3$ | $x(3,7) = 12$ | $y(3,7) = -8$ |
| $x_u(9,11) = 4$ | $y_u(9,11) = 5$ | $x_u(3,7) = -12$ | $y_u(3,7) = 6$ |
| $x_v(9,11) = -6$ | $y_v(9,11) = 7$ | $x_v(3,7) = 4$ | $y_v(3,7) = 14$ |
| $z(-3,3) = 16$ | $z(9,11) = 17$ | $z(3,7) = 18$ | $z(12,-8) = 19$ |
| $z_x(-3,3) = -2$ | $z_y(-3,3) = 9$ | $z_x(9,11) = 2$ | $z_y(9,11) = -9$ |
| $z_x(3,7) = 5$ | $z_y(3,7) = 13$ | $z_x(12,-8) = -5$ | $z_y(12,-8) = -7$ |

Fill in the following boxes with numbers using this table. No questions asked!

$\left.\frac{\partial x}{\partial s}\right|_{(s,t)=(3,7)} =$ 54

$\left.\frac{\partial y}{\partial t}\right|_{(s,t)=(3,7)} =$ -11

$\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(3,7)} =$ 369

$\left.\frac{\partial z}{\partial t}\right|_{(s,t)=(3,7)} =$ -151

The linearization of $z$ as a function of $x$ and $y$ at the point $(x, y) = (12, -8)$ is

$L(x, y) =$ 23 $+$ -5 $x +$ -7 $y$

The linearization of $z$ as a function of $s$ and $t$ at the point $(s, t) = (3, 7)$ is $(s, t) = (3, 7)$ is

$L(s, t) =$ -34 $+$ 369 $s +$ -151 $t$

Grading: Each correctly filled box is 1 point. Grader: `melis.gezer@bilkent.edu.tr`

**Solution:**

$$\left.\frac{\partial x}{\partial s}\right|_{(s,t)=(3,7)} = \left.\frac{\partial x}{\partial u}\right|_{(u,v)=(9,11)} \left.\frac{\partial u}{\partial s}\right|_{(s,t)=(3,7)} + \left.\frac{\partial x}{\partial v}\right|_{(u,v)=(9,11)} \left.\frac{\partial v}{\partial s}\right|_{(s,t)=(3,7)} = (4)(12) + (-6)(-1) = 54$$

$$\left.\frac{\partial x}{\partial t}\right|_{(s,t)=(3,7)} = \left.\frac{\partial x}{\partial u}\right|_{(u,v)=(9,11)} \left.\frac{\partial u}{\partial t}\right|_{(s,t)=(3,7)} + \left.\frac{\partial x}{\partial v}\right|_{(u,v)=(9,11)} \left.\frac{\partial v}{\partial t}\right|_{(s,t)=(3,7)} = (4)(2) + (-6)(-3) = 26$$

$$\left.\frac{\partial y}{\partial s}\right|_{(s,t)=(3,7)} = \left.\frac{\partial y}{\partial u}\right|_{(u,v)=(9,11)} \left.\frac{\partial u}{\partial s}\right|_{(s,t)=(3,7)} + \left.\frac{\partial y}{\partial v}\right|_{(u,v)=(9,11)} \left.\frac{\partial v}{\partial s}\right|_{(s,t)=(3,7)} = (5)(12) + (7)(-1) = 53$$

$$\left.\frac{\partial y}{\partial t}\right|_{(s,t)=(3,7)} = \left.\frac{\partial y}{\partial u}\right|_{(u,v)=(9,11)} \left.\frac{\partial u}{\partial t}\right|_{(s,t)=(3,7)} + \left.\frac{\partial y}{\partial v}\right|_{(u,v)=(9,11)} \left.\frac{\partial v}{\partial t}\right|_{(s,t)=(3,7)} = (5)(2) + (7)(-3) = -11$$

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(3,7)} = \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(-3,3)} \left.\frac{\partial x}{\partial s}\right|_{(s,t)=(3,7)} + \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(-3,3)} \left.\frac{\partial y}{\partial s}\right|_{(s,t)=(3,7)} = (-2)(54) + (9)(53) = 369$$

$$\left.\frac{\partial z}{\partial t}\right|_{(s,t)=(3,7)} = \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(-3,3)} \left.\frac{\partial x}{\partial t}\right|_{(s,t)=(3,7)} + \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(-3,3)} \left.\frac{\partial y}{\partial t}\right|_{(s,t)=(3,7)} = (-2)(26)+(9)(-11) = -151$$

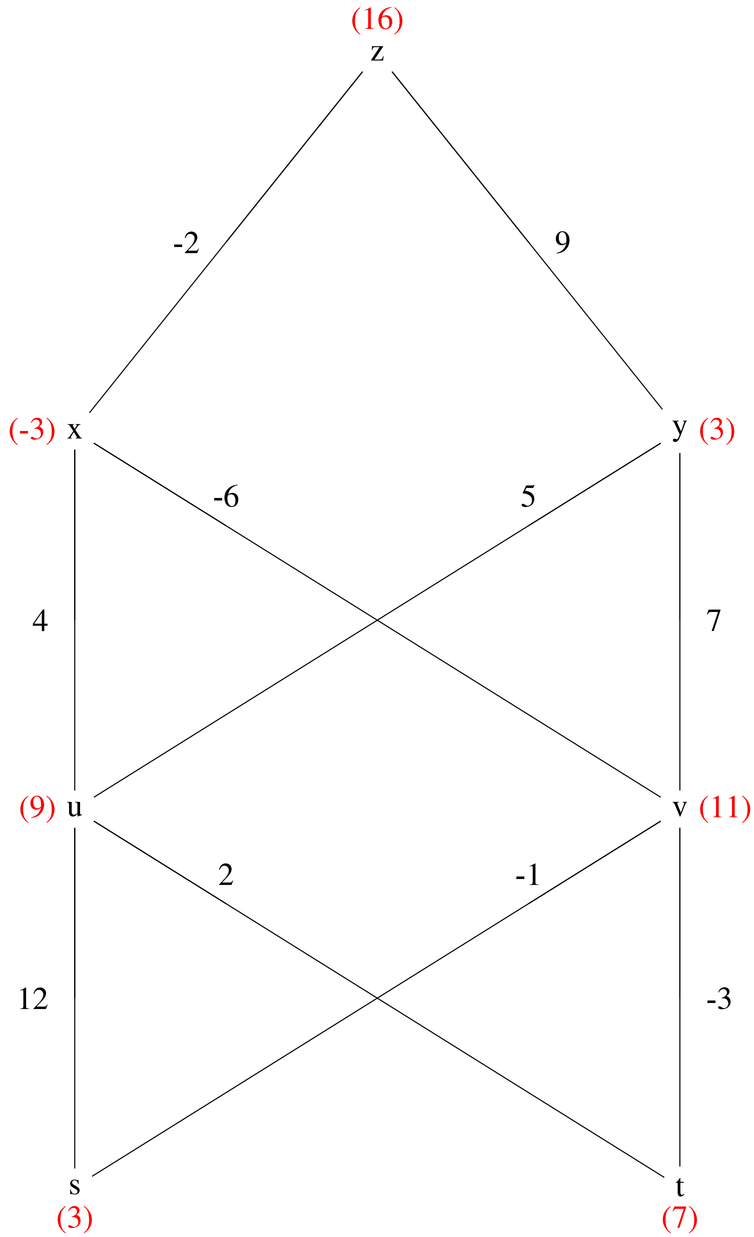
The linearization of $z$ as a function of $x$ and $y$ at the point $(x, y) = (12, -8)$ is

$$\begin{aligned} L(x,y) &= \left( z\Big|_{(x,y)=(12,-8)} \right) + \left( \left.\frac{\partial z}{\partial x}\right|_{(x,y)=(12,-8)} \right) (x-12) + \left( \left.\frac{\partial z}{\partial y}\right|_{(x,y)=(12,-8)} \right) (y+8) \\ &= (19) + (-5)(x-12) + (-7)(y+8) \\ &= 23 - 5x - 7y. \end{aligned}$$

The linearization of $z$ as a function of $s$ and $t$ at the point $(s, t) = (3, 7)$ is

$$\begin{aligned} L(s,t) &= \left( z\Big|_{(s,t)=(3,7)} \right) + \left( \left.\frac{\partial z}{\partial s}\right|_{(s,t)=(3,7)} \right) (s-3) + \left( \left.\frac{\partial z}{\partial t}\right|_{(s,t)=(3,7)} \right) (t-7) \\ &= (16) + (369)\ (s-3) + (-139)\ (t-7) \\ &= -34 + 369s - 151t. \end{aligned}$$

An alternate approach is on next page.




The number in parenthesis near a variable is its value at the relevant point. The number adjacent to a line is the value of the relevant partial derivative at that point.

Now for example to calculate $\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(3,7)}$ follow all the paths from $z$ to $s$ and for each such path multiply the numbers you see adjacent to the lines you follow. Then add all these numbers.

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t)=(3,7)} = (-2)(4)(12) + (-2)(-6)(-1) + (9)(5)(12) + (9)(7)(-1) = -96 - 12 + 540 - 63 = 369.$$