Quiz # 01
Math 102 Section 11 Calculus II
4 February 2025, Tuesday
Instructor: Ali Sinan Sertöz
Solution Key

Bilkent University

Q-1) Find the limit of the sequence $(a_n)_{n=1}^{\infty}$ given recursively as

$$a_1 = 1, \text{ and } a_n = \frac{n^2-1}{n^2} a_{n-1} \text{ for } n > 1.$$

Find $\lim_{n\to\infty} a_n$ if it exists. If it does not exist, explain why.

Hint: You may start by proving by induction that $a_n = \frac{n+1}{2n}$ for $n \geq 1$.

Grading: 10 points

Solution: Grader: `emre.baran@ug.bilkent.edu.tr`

We follow the hint and claim that $a_n = \frac{n+1}{2n}$ for $n \geq 1$. We prove this by induction.

When $n = 1$, the claim says that $a_1 = 1$ which is the given definition so the claim holds for $n = 1$.

Now we assume that $a_n = \frac{n+1}{2n}$, for some $n \geq 1$ and check the claim for $n + 1$.

$$\begin{aligned} a_{n+1} &= \frac{(n+1)^2-1}{(n+1)^2} a_n \quad \text{by the given definition of the sequence} \\ &= \frac{(n+1)^2-1}{(n+1)^2} \frac{n+1}{2n} \quad \text{by the induction hypothesis} \\ &= \frac{(n+1)+1}{2(n+1)} \quad \text{after simplification} \end{aligned}$$

which shows that the claim holds for $n + 1$ if it holds for n.

This completes the proof by induction. Now we can calculate the limit.

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n+1}{2n} \\ &= \frac{1}{2}. \end{aligned}$$

Thus we found that the required limit is $\frac{1}{2}$.