Date: 23 July 2004, Friday
Instructor: Ali Sinan Sertöz
Time: 13:30-15:30

Math 112 Intermediate Calculus II – Final Exam – Solutions

Q-1) Evaluate the integral $\int t\sqrt{1+t^2}\arctan\sqrt{1+t^2}\,dt$.

Solution: Put $x = \sqrt{1+t^2}$, then $\int t\sqrt{1+t^2}\arctan\sqrt{1+t^2}\;\;dt = \int x^2\arctan x\;\;dx$. Using by-parts with $u=\arctan x$ and $dv = x^2dx$ we get $\int x^2\arctan x\;dx = \frac{1}{3}x^3\arctan x - \frac{1}{3}\int\frac{x^3}{1+x^2}\,dx = \frac{1}{3}x^3\arctan x - \frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)\,dx = \frac{1}{3}x^3\arctan x - \frac{1}{6}x^2+\frac{1}{6}\ln(1+x^2)+C.$

Finally putting back the original substitution gives

$$\int t\sqrt{1+t^2}\arctan\sqrt{1+t^2}\,dt = \frac{1}{3}(1+t^2)^{3/2}\arctan\sqrt{1+t^2} - \frac{1}{6}(1+t^2)+\frac{1}{6}\ln(2+t^2)+C.$$

Q-2) Find the the <u>interval</u> of convergence of the power series $\sum_{n=0}^{\infty}\frac{2^n(n!)^2}{(2n)!}(x-5)^n$.

Solution: Let $a_n = \frac{2^n(n!)^2}{(2n)!}(x-5)^n$. Using ratio test we get $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2}|x-5|$. So the series converges absolutely for $3<x<7$.

Now we check the end points. When $x=7$ or $x=3$, we have $|a_n| = \frac{4^n(n!)^2}{(2n)!}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4n^2+8n+4}{4n^2+6n+2} > 1.$$

Then $|a_{n+1}| > |a_n| > \cdots > |a_0| = 1$. So in particular $\lim_{n\to\infty} a_n \neq 0$, so the series diverges at the end points.

Hence the interval of convergence is $3<x<7$.

Q-3) Find a parametric equation for the tangent line of the curve of intersection of the two surfaces $x^3+y^2-xz^3=6$ and $y^3-xyz-2x^2=8$ at the point $(1,2,-1)$.

Solution: Let $f = x^3+y^2-xz^3-6$, $g=y^3-xyz-2x^2-8$.

$\nabla f = (3x^2-z^3, 2y, -3xz^2)$, $\nabla g = (-yz-4x, 3y^2-xz, -xy)$.

$\nabla f(1,2,-1) = (4,4,-3)$, $\nabla g(1,2,-1) = (-2,13,-2)$.

$$\nabla f(1,2,-1) \times \nabla g(1,2,-1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 4 & -3 \\ -2 & 13 & -2 \end{vmatrix} = (31, 14, 60).$$

A parametric equation for the tangent line in question is $L(t) = (1,2,-1) + (31,14,60)t$, $t \in \mathbb{R}$.

Note that this line is in the intersection of the tangent planes of the surfaces $f = 0$ and $g = 0$ at the given point, so you can write the equations of their tangent planes and that would constitute another description of this tangent line.

Q-4) Find the critical points of $f(x,y) = x^4 + 4xy + y^4$ and decide if each critical point is a local min/max or a saddle point. Find global extreme points and values of $f(x,y)$, if they exist.

Solution: Solving $f_x = 4x^3 + 4y = 0$ and $f_y = 4x + 4y^3 = 0$ we find that the critical points are $(0,0)$, $(1,-1)$ and $(-1,1)$.

To apply the second derivative test we calculate $f_{xx} = 12x^2$, $f_{yy} = 12y^2$, $f_{xy} = 4$ and $\Delta = 144x^2y^2 - 16$.

We then find that $(0,0)$ is a saddle point, and the other two critical points are local min points.

Since f is bounded from below, it must have a minimum point and it must be at one of these local min points. Since $f(-1,1) = f(1,-1) = -2$, this is the global min value for the function. f clearly is unbounded from above.

Q-5) Find the maximum value of the function $f(x,y,z) = xyz$ subject to the conditions $x + y^2 + z^3 = 1188$, $x > 0$, $y > 0$ and $z > 0$.

Solution: Let $g = x + y^2 + z^3 - 1188$. We use Lagrange multipliers method.

The system $\nabla f = \lambda \nabla g$ and $g = 0$ is to be solved.

(1)... $yz = \lambda$
(2)... $xz = 2\lambda y$
(3)... $xy = 3\lambda z^2$
(4)... $x + y^2 + z^3 - 1188 = 0.$

(1) and (2) $\Rightarrow$ $xz = 2y^2z$. Since $z > 0$, we can cancel z to obtain $y^2 = x/2$.

(1) and (3) similarly imply $z^3 = x/3$.

Putting these into (4) we get $x = 648$. This then gives $y = 18$ and $z = 6$.

The surface $x + y^2 + z^3 - 1188 = 0$ with $x, y, z \geq 0$ is a closed and bounded set in $\mathbb{R}^3$. Since f is continuous, it must have a minimum and maximum on this surface. Since the minimum of f is clearly 0, the above critical point must give the global maximum.

Therefore the maximum value of f is $648 \cdot 18 \cdot 6 = 69984$.
