Math 112 Intermediate Calculus II – Midterm Exam II – Solutions

Q-1) i) Find \( \lim_{x \to 0} \ln(1 + x)^{\ln(1+x)} \).

Solution-i: If \( L = \lim_{x \to 0} \ln(1 + x)^{\ln(1+x)} \), then
\[
\ln L = \lim_{x \to 0} \ln(1 + x) \cdot \ln(1 + x) = \lim_{x \to 0} \frac{\ln(1 + x)}{\ln(1 + x)} = \lim_{x \to 0} \ln(1 + x) = 0
\]
after applying L’Hopital’s rule. Hence \( L = 1 \).

ii) Find \( \lim_{n \to \infty} \frac{n!}{n^n} \).

Solution-ii: \( 0 < \frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots n}{n \cdot n \cdot n \cdots n} < \frac{1}{n} \). Then by the sandwich theorem \( \lim_{n \to \infty} \frac{n!}{n^n} = 0 \)

Q-2) Find the sum \( \sum_{n=1}^{\infty} \frac{5n + 6}{n^3 + 3n^2 + 2n} \).

Solution: By the partial fractions method we find that \( \frac{5n + 6}{n^3 + 3n^2 + 2n} = \frac{3}{n} - \frac{1}{n + 1} - \frac{2}{n + 2} \). Adding the series telescopically we find that the partial sums are of the form \( S_n = 4 - \frac{1}{n + 1} - \frac{2}{n + 2} \). Hence the sum is \( \lim_{n \to \infty} S_n = 4 \).

Q-3) Are these series conditionally convergent, absolutely convergent or divergent?

i) \( \sum_{n=1}^{\infty} (-1)^n \ln \left(1 + \frac{1}{n}\right) \).

Solution-i: Since \( \ln \left(1 + \frac{1}{n}\right) \) decreases to zero, this series converges by the alternating series test. However, limit-comparing with the harmonic series implies that the series of absolute values diverges. Hence this series conditionally converges.

ii) \( \sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(n!)^2 5^n} \).

Solution-ii: Using the ratio test we see that this series converges absolutely.

Q-4) Find the interval of convergence, i.e. check also the end points, for the power series \( \sum_{n=1}^{\infty} n^{1/n} x^n \).

Solution: Letting \( a_n = n^{1/n} |x|^n \), and applying the n-th root test we see that \( \lim_{n \to \infty} (a_n)^{1/n} = |x| \), so the series absolutely converges for all \( |x| < 1 \). When \( x = \pm 1 \), the absolute value of the general term is \( n^{1/n} \) which converges to 1 as \( n \) goes to infinity. Hence the series diverges at the end point by the divergence test.
Q-5) Find \( \lim_{x \to 0} \frac{(\ln \frac{1+x}{1-x}) - 2x}{(1 - e^x)(x \sin x)} \).

Solution:

\[
\left( \ln \frac{1+x}{1-x} \right) - 2x = \ln(1+x) - \ln(1-x) - 2x = \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \right) - \left( x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right) - 2x = \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots.
\]

\[(1 - e^x)(x \sin x) = \left( -x - \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots \right) \left( x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots \right) = -x^3 - \frac{x^4}{2!} + \cdots.
\]

Then \( \lim_{x \to 0} \frac{(\ln \frac{1+x}{1-x}) - 2x}{(1 - e^x)(x \sin x)} = \lim_{x \to 0} \frac{\frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots}{-x^3 - \frac{x^4}{2!} + \cdots} = -\frac{2}{3}. \)