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Math 112 Intermediate Calculus II – Midterm Exam II – Solutions

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Q-1) i) Find $\lim_{x\to 0} [\ln(1+x)]^{\ln(1+x)}$. **ii)** Find $\lim_{n\to\infty} \frac{n!}{n^n}$.

Solution-i: If $L = \lim_{x\to 0} [\ln(1+x)]^{\ln(1+x)}$, then $\ln L = \lim_{x\to 0} \ln(1+x) \cdot \ln\ln(1+x) = \lim_{x\to 0} \frac{\ln(\ln(1+x))}{\frac{1}{\ln(1+x)}} = \lim_{x\to 0} \ln(1+x) = 0$ after applying L'Hopital's rule. Hence $L = 1$.

Solution-ii: $0 < \frac{n!}{n^n} = \frac{1}{n} \frac{2\cdot 3\cdot 4 \cdots n}{n\cdot n\cdot n \cdots n} < \frac{1}{n}$. Then by the sandwich theorem $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

Q-2) Find the sum $\sum_{n=1}^{\infty} \frac{5n+6}{n^3+3\,n^2+2\,n}$.

Solution:

By the partial fractions method we find that $\frac{5n+6}{n^3+3\,n^2+2\,n} = \frac{3}{n} - \frac{1}{n+1} - \frac{2}{n+2}$. Adding the series telescopingly we find that the partial sums are of the form $S_n = 4 - \frac{3}{n+1} - \frac{2}{n+2}$. Hence the sum is $\lim_{n\to\infty} S_n = 4$.

Q-3) Are these series conditionally convergent, absolutely convergent or divergent?

i) $\sum_{n=1}^{\infty} (-1)^n \ln\left(1+\frac{1}{n}\right)$. **ii)** $\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{(n!)^2\, 5^n}$.

Solution-i: Since $\ln\left(1+\frac{1}{n}\right)$ decreases to zero, this series converges by the alternating series test. However, limit-comparing with the harmonic series implies that the series of absolute values diverges. Hence this series conditionally converges.

Solution-ii: Using the ratio test we see that this series converges absolutely.

Q-4) Find the interval of convergence, i.e. check also the end points, for the power series $\sum_{n=1}^{\infty} n^{1/n}\, x^n$.

Solution:

Letting $a_n = n^{1/n}\,|x|^n$, and applying the n-th root test we see that $\lim_{n\to\infty} (a_n)^{1/n} = |x|$, so the series absolutely converges for all $|x| < 1$. When $x = \pm 1$, the absolute value of the general term is $n^{1/n}$ which converges to 1 as n goes to infinity. Hence the series diverges at the end point by the divergence test.

Q-5) Find $\lim_{x\to 0} \frac{\left(\ln \frac{1+x}{1-x}\right) - 2x}{(1-e^x)(x\sin x)}$.

Solution:

$$\left(\ln \frac{1+x}{1-x}\right) - 2x = \ln(1+x) - \ln(1-x) - 2x$$

$$= (x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots) - (x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots) - 2x = \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots.$$

$$(1-e^x)(x\sin x) = (-x - \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots)(x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \cdots) = -x^3 - \frac{x^4}{2!} + \cdots.$$

Then $\lim_{x\to 0} \frac{\left(\ln \frac{1+x}{1-x}\right) - 2x}{(1-e^x)(x\sin x)} = \lim_{x\to 0} \frac{\frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots}{-x^3 - \frac{x^4}{2!} + \cdots} = -\frac{2}{3}.$
