Oct 20, 2003

MATH 113 HOMEWORK 2
SOLUTION MANUAL

1) Find a formula for each of the following expressions and prove your formula using induction.
i) $1^2 + 2^2 + \cdots + n^2$.
Solution. We have $(k+1)^3 - k^3 = 3k^2 + 3k + 1$. Writing this for $k = 1, 2, 3 \cdots, n$ and

$$\begin{aligned}
2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\
3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\
4^3 - 3^3 &= 3 \cdot 3^2 + 3 \cdot 3 + 1 \\
\cdots & \quad \cdots \\
(n+1)^3 - n^3 &= 3 \cdot n^2 + 3 \cdot n + 1
\end{aligned}$$

and adding we get

$$(n+1)^3 - 1 = 3 \sum_{k=1}^{n} k^2 + 3 \sum_{k=1}^{n} +n$$

and solving this for $\sum_{k=1}^{n} k^2$ we obtain

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \qquad (*).$$

Now we prove this by induction for all integers $n \geq 1$.
For $n = 1$, both sides are equal to 1. Assume the formula is true for some integer $n \geq 1$, i.e. assume that $1^2 + 2^2 + \cdots n^2 = \frac{n(n+1)(2n+1)}{6}$. Add $(n+1)^2$ to both sides. We get

$$\begin{aligned}
1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)[n(2n+1) + 6(n+1)]}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)[(n+1)+1][2(n+1)+1]}{6}
\end{aligned}$$

So the formula is true for $n+1$. Thus by induction the formula (*) is true for all integers $n \geq 1$.

ii) $1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^{n-1}n^2$.
Solution. Let $f(n) = 1 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2$. We try a few values of n. We see that

$$f(1) = 1,\ f(2) = -3,\ f(3) = 6,\ f(4) = -10,\ f(5) = 15,$$

and these are numerically same as

$$1,\ 1+2=3,\ 1+2+3=6,\ 1+2+3+4=10,\ 1+2+3+4+5=15$$

but signs alternate. So we guess that

$$f(n) = 1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2} \text{ for all } n \geq 1 \qquad (**).$$

Now we prove (**) by induction.
For $n = 1$, $f(1) = 1$ and $(-1)^{n-1}\frac{n(n+1)}{2} = 1$, so the formula is true for $n = 1$. Now assume (**) is true for some integer $n \geq 1$, i.e. assume

$$f(n) = 1^2 - 2^2 + \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}\frac{n(n+1)}{2}.$$

Add $(-1)^n(n+1)^2$ to both sides

$$\begin{aligned} f(n+1) = 1^2 - 2^2 + \cdots + (-1)^{n-1}n^2 + (-1)^n(n+1)^2 &= (-1)^{n-1}\frac{n(n+1)}{2} + (-1)^n(n+1)^2 \\ &= (-1)^{n-1}(n+1)\left(\frac{n}{2} - (n+1)\right) \\ &= (-1)^{n-1}(n+1)\frac{-n-2}{2} \\ &= (-1)^n\frac{(n+1)(n+2)}{2}. \end{aligned}$$

So the formula is true for $n+1$. Thus by mathematical induction the formula (**) is true for all integers $n \geq 1$.

iii) $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)$.

Solution. Let

$$g(n) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) \text{ for } n \geq 2.$$

Factoring each parenthesis

$$\begin{aligned} g(n) &= \left(1 - \frac{1}{2}\right)\left(1 + \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{4}\right)\cdots\left(1 - \frac{1}{n}\right)\left(1 + \frac{1}{n}\right) \\ &= \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdots\frac{n-1}{n}\cdot\frac{n+1}{n} \\ &= \frac{n+1}{2n} \end{aligned}$$

So we claim that

$$g(n) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \text{ for } n \geq 2 \qquad (***)$$

For $n = 2$, $1 - \frac{1}{2^2} = \frac{3}{4}$ and $\frac{n+1}{2n} = \frac{3}{4}$, so the claim is true for $n = 2$. Assume the formula (***) is true for some integer $n \geq 2$, i.e.,

$$g(n) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n},$$

multiplying both sides by $1 - \frac{1}{(n+1)^2}$,

$$\begin{aligned} g(n+1) &= g(n)\left(1 - \frac{1}{(n+1)^2}\right) = \frac{n+1}{2n}\left(1 - \frac{1}{(n+1)^2}\right) \\ &= \frac{n+1}{2n} \cdot \frac{n^2+2n}{(n+1)^2} = \frac{n+2}{2(n+1)}. \end{aligned}$$

So the formula (***) is true for $n+1$. Thus by induction the formula (***) is true for all integers $n \geq 2$.

2) Solve Exercises 6 and 7 on page 64.
Solution of Exercise 6. Let a and b be integers such that $a < b$, and f be a nonnegative real valued function defined on $[a, b]$. Let S be the set

$$S = \{(x, y) : a \leq x \leq b,\ 0 < y \leq f(x)\}.$$

We are asked to show that the number of lattice points in S is given by

$$\sum_{n=a}^{b} [f(n)].$$

(Here $[x]$ denotes the greatest integer $\leq x$.)
Note that S does not contain the part of the x-axis in the ordinate set of f.
Given any positive real number y, $[y]$ is the number of integers k such that $0 < k \leq y$. If $y = 0$, then, there is no integer k such that $0 < k \leq y$ and $[y]$=0. So if $y \geq 0$, then $[y]$ is the number of integers k such that $0 < k \leq y$. Now if n is an integer, taking $y = f(n)$, we have that $[f(n)]$ is the number of integers k such that $0 < k \leq f(n)$, i.e. the number of lattice points on the half open segment $\{(x, y) : x = n, 0 < y \leq f(n)\}$. As n changes through the integer values in the interval $[a, b]$ we get all the lattice points in the set S.

Solution of Exercise 7. Let a and b be positive integers with no common factors. We are asked to show that

$$\sum_{n=1}^{b-1} \left[\frac{na}{b}\right] = \frac{(a-1)(b-1)}{2}.$$

For $b = 1$, define the sum on the left hand side as 0. So assume that $b \geq 2$.
a) Consider the set

$$S = \{(x, y) : 1 \leq x \leq b-1,\ 0 < y \leq \frac{a}{b}x\}$$

This is the set S in the previous problem with $f(x) = \frac{a}{b}x$. According to the previous problem the number of lattice points in S is $\sum_{n=1}^{b-1} \left[\frac{a}{b}n\right]$.
Now let us count these lattice points some other way. Consider the line segment $y = \frac{a}{b}x$, for

$1 \le x \le b-1$. On this line segment there are no lattice points. For if (x,y) is a point on this segment, then it is a lattice point if and only if $x = n$ is an integer (where $1 \le n \le b-1$) and $y = \frac{a}{b}n$ is an integer, that is b divides an. The fact that b and a have no common factors implies that b must divide n. But since $1 \le b \le n-1$, this is impossible. Consider the rectangle $R = \{(x,y) : 0 \le x \le b,\ 0 \le y \le a\}$. In the interior of this rectangle there are $(a-1)(b-1)$ lattice points and none of them are on the South West-North East diagonal. Then the lattice points inside the interior of R which are below this diagonal are exactly those which are inside S and the number of them are $\frac{(a-1)(b-1)}{2}$.

b) Let $C = \sum_{n=1}^{b-1} \left[\frac{na}{b}\right]$. Change the summation index to $k = b-n$. Then

$$\begin{aligned}
C &= \sum_{n=1}^{b-1}\left[\frac{na}{b}\right] = \sum_{k=b-1}^{1}\left[\frac{(b-k)a}{b}\right] = \sum_{k=1}^{b-1}\left[a - \frac{ka}{b}\right] \\
&= \sum_{k=1}^{b-1}\left(a + \left[-\frac{ka}{b}\right]\right) \quad \text{(by exercise 4.a) on page 64.)} \\
&= \sum_{k=1}^{b-1}\left(a - \left[\frac{ka}{b}\right] - 1\right) \quad \text{(by exercise 4.b) on page 64.)} \\
&= (a-1)(b-1) - \underbrace{\sum_{k=1}^{b-1}\left[\frac{ka}{b}\right]}_{C}
\end{aligned}$$

So

$$C = (a-1)(b-1) - C \Rightarrow C = \frac{(a-1)(b-1)}{2}.$$

3) Evaluate the integral $\int_{-2}^{5} |x^2 - 2x| dx$.

Solution: First we examine the sign of $f(x) = x^2 - 2x$. It is easy to see that $f(x) = 0$ when $x = 0$ or when $x = 2$. It the follows that $|x^2 - 2x| = \begin{cases} x^2 - 2x & \text{if } x \notin (0,2), \\ 2x - x^2 & \text{if } x \in [0,2]. \end{cases}$

We can now easily evaluate our integral

$$\begin{aligned}
\int_{-2}^{5} |x^2 - 2x|\, dx &= \int_{-2}^{0}(x^2 - 2x)dx + \int_{0}^{2}(2x - x^2)dx + \int_{2}^{5}(x^2 - 2x)dx \\
&= \left(\frac{x^3}{3} - x^2\Big|_{-2}^{0}\right) + \left(x^2 - \frac{x^3}{3}\Big|_{0}^{2}\right) + \left(\frac{x^3}{3} - x^2\Big|_{2}^{5}\right) \\
&= \frac{20}{3} + \frac{4}{3} + 18 \\
&= 26.
\end{aligned}$$

4) Solve Exercise 14 on page 114: A napkin-ring is formed by drilling a cylindrical hole symmetrically through the center of a solid sphere. If the length of the hole is $2h$, prove that the volume of the napkin-ring is $\pi a h^3$, where a is a rational number.

Solution: Assume that the solid sphere is given by the equation $x^2 + y^2 + z^2 = r^2$, where r is its radius. With these coordinates assume that the cylindrical hole that is drilled out is expressed by the equation $x^2 + y^2 = c^2$ for some positive constant c. Now assume that we cut this napkin-ring by the yz-plane, or equivalently by the $x = 0$ plane. The resulting picture is depicted in the following figure. If however the napkin-ring is cut by a plane perpendicular to the yz-plane along the line AC, the slice obtained would look like a ring, consisting of a disk of radius AC out of which which a disk of radius AB is cut off. The area of this ring is $\pi(AC^2 - AB^2)$. So we set out to write this area explicitly:

Since the sphere $x^2 + y^2 + z^2 = r^2$ is cut off by the plane $x = 0$, the resulting circle of the figure has the form $y^2 + z^2 = r^2$.

Since the point (c, h) is on this circle, we have $c^2 = r^2 - h^2$. Observe that $c = AB$.

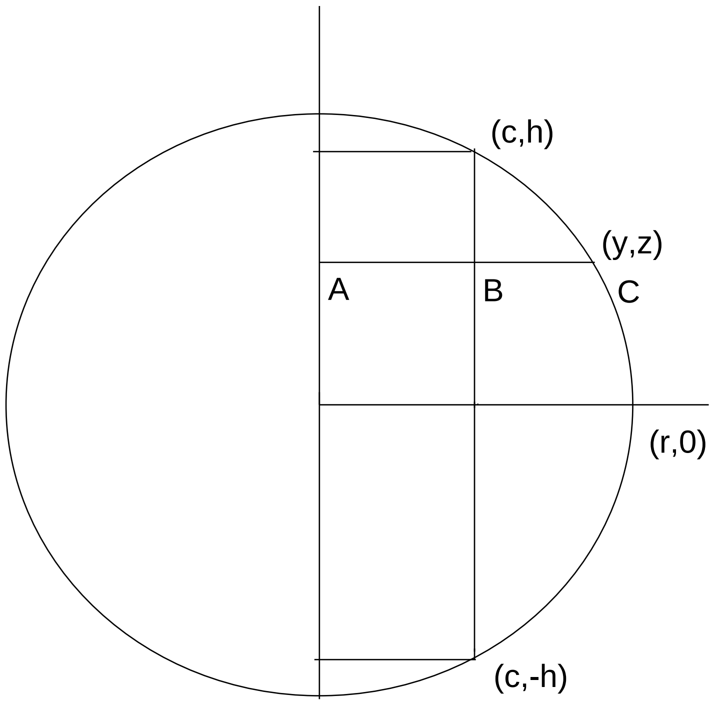
Since the point (y, z) is on the circle, we have as above $y^2 = r^2 - z^2$. Observe again that $y = AC$.

Thus the area of the slice is $\pi(AC^2 - AB^2) = \pi(h^2 - z^2)$.

To find the volume we have to add/integrate all these areas as z changes from $-h$ to h.

$$\begin{aligned} \text{Volume} &= \int_{-h}^{h} \pi(h^2 - z^2)dz \\ &= \pi \left(h^2 z - \frac{z^3}{3} \Big|_{-h}^{h} \right) \\ &= \frac{4}{3}\pi h^3, \end{aligned}$$

as claimed. The surprising thing about this result is that the volume of the napkin-ring is independent of the radius of the solid sphere out of which it is cut off.



The figure for the napkin-ring problem.

27 October 2003 Monday.