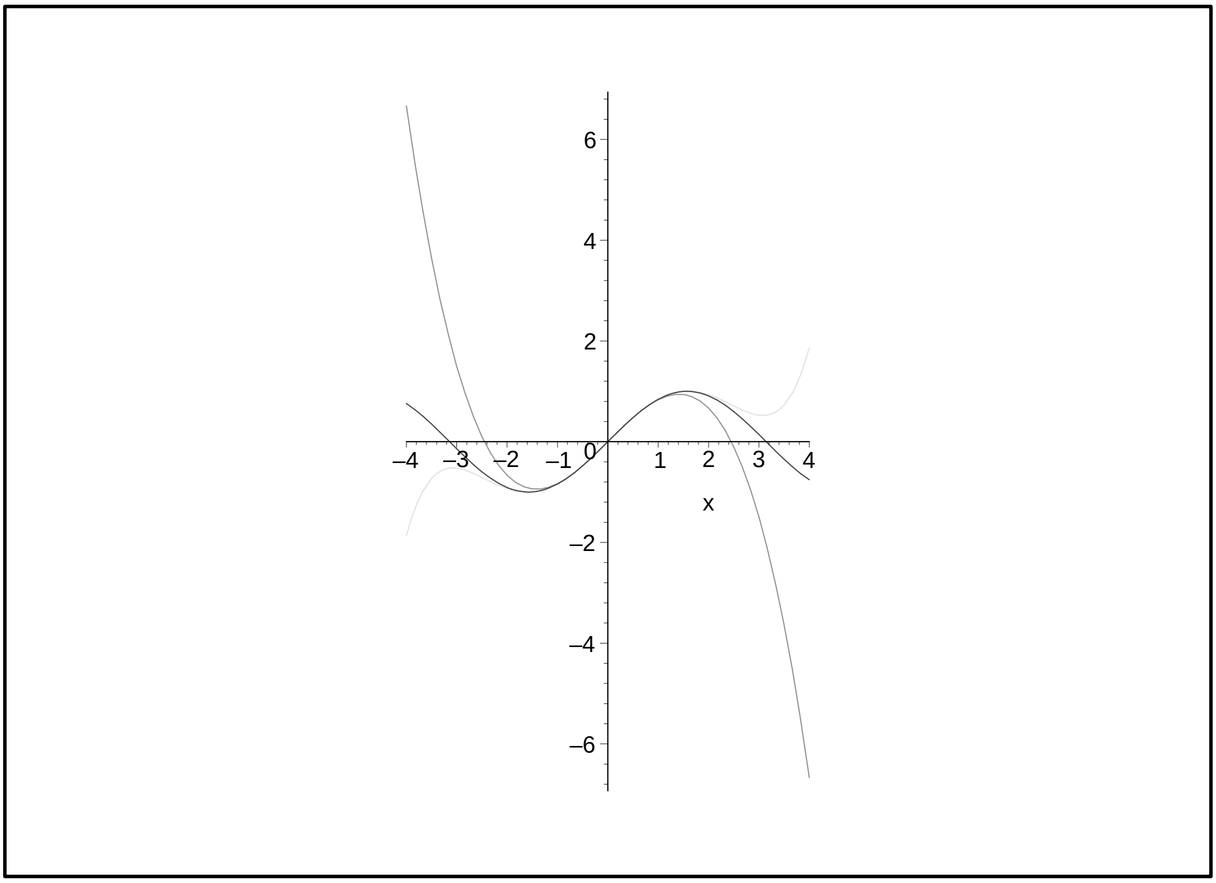
MATH 113 Solutions manual for Homework VI

1. Page 278 Exercise 1:
Draw graphs of the Taylor Polynomials $T_3(\sin x) = x - x^3/3!$ and $T_5(\sin x) = x - x^3/3! + x^5/5!$. Pay careful attention to the points where the curves cross the x-axis. Compare these graphs with that of $f(x) = \sin x$.

Solution:

The graphs of $T_3(\sin x)$, $T_5(\sin x)$ and $\sin x$ are given below:



2. Page 278 Exercise 9:
Let α be a real number. Show that

$$T_n[(1+x)^\alpha] = \sum_{k=0}^{n} \left(\begin{array}{c} \alpha \\ k \end{array} \right) x^k \quad \text{where} \quad \left(\begin{array}{c} \alpha \\ k \end{array} \right) = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

Solution:
Let $f(x) = (1+x)^\alpha$. By taking successive derivatives of f and evaluating at $x = 0$ we find that:

$f(x) = (1+x)^\alpha$, $f(0) = 1$

$f'(x) = \alpha(1+x)^{\alpha-1}$, $f'(0) = \alpha$,

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$, $f''(0) = \alpha(\alpha-1)$,

$\vdots$ $f^{(k)}(x) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+x)^{\alpha-k}$, $f^{(k)}(0) = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$.

The general term of the Taylor polynomial around 0 then becomes

$\dfrac{f^{(k)}(0)}{k!}$, which is precisely $\left(\begin{array}{c} \alpha \\ k \end{array} \right)$.

3-a. Find $T_9(\tan x; 0)$.

Solution:
$T_9(\tan x; 0) = x + \dfrac{1}{3}\, x^3 + \dfrac{2}{15}\, x^5 + \dfrac{17}{315}\, x^7 + \dfrac{62}{2835}\, x^9$.

3-b. Find $T_n(\sin x; \pi/3)$.

Solution:

$$T_n(\sin x; \pi/3) = \sum_{k=0}^{n} \frac{\epsilon_k}{2 \cdot k!} \left(x - \frac{\pi}{3}\right)^k, \quad \text{where} \quad \epsilon_k = \begin{cases} \sqrt{3} & \text{if } \; k \equiv 0 \bmod 4, \\ 1 & \text{if } \; k \equiv 1 \bmod 4, \\ -\sqrt{3} & \text{if } \; k \equiv 2 \bmod 4, \\ -1 & \text{if } \; k \equiv 3 \bmod 4. \end{cases}$$
