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Math 113 Calculus – Final Exam
Solutions

Q-1) Find the derivatives of the following functions:

a) $y = x^{(x^x)}$. **b)** $y = (\sin x)^{\ln x}$. **c)** $y = x^\pi + \pi^x + \pi^\pi + x^x$.

Solution-a)

$$\begin{aligned}
\ln y &= x^x \ln x \\
\ln \ln y &= x \ln x + \ln \ln x. \\
\frac{y'}{y \ln y} &= \ln x + 1 + \frac{1}{x \ln x}. \\
y' &= (x^{(x^x)})(x^x \ln x)(\ln x + 1 + \frac{1}{x \ln x}).
\end{aligned}$$

Solution-b)

$$\begin{aligned}
\ln y &= (\ln x)(\ln \sin x). \\
\frac{y'}{y} &= \frac{\ln \sin x}{x} + \frac{\ln x}{\sin x} \cos x. \\
y' &= (\sin x)^{\ln x} \left(\frac{\ln \sin x}{x} + \frac{\ln x}{\sin x} \cos x \right).
\end{aligned}$$

Solution-c)

$$y' = \pi x^{\pi - 1} + (\ln \pi)\pi^x + 0 + (\ln x + 1)x^x.$$

Q-2) Evaluate the integral $\int \sqrt{1 + x^2}\, dx$.

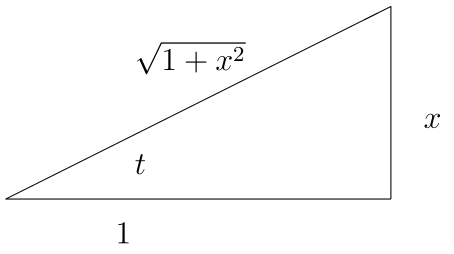
Solution: Put $x = \tan t$, $dx = \sec^2 t\, dt$. The integral then becomes

$$\int \sqrt{1 + x^2}\, dx = \int \sec^3 t\, dt.$$

Using integration by parts with $u = \sec t$ we get

$$\begin{aligned}
\int \sec^3 t\, dt &= \sec t \tan t - \int \sec t \tan^2 t\, dt \\
&= \sec t \tan t - \int \sec t(\sec^2 t - 1)t\, dt \\
&= \sec t \tan t - \int \sec^3 t\, dt + \int \sec t\, dt \\
\int \sec^3 t\, dt &= \frac{1}{2} \sec t \tan t + \frac{1}{2} \ln|\sec t + \tan t| + C.
\end{aligned}$$

Now we have to convert this to a function of x. For this we use the initial substitution $x = \tan t$ together with the following triangle.



Here it is clear that $\sec t = \sqrt{1+x^2}$ and $\tan t = x$. Putting these in we find finally

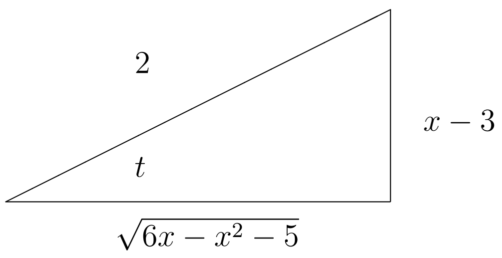
$$\int \sqrt{1+x^2}\, dx = \frac{1}{2} x\sqrt{1+x^2} + \frac{1}{2} \ln |\sqrt{1+x^2} + x| + C.$$

Q-3) Evaluate the integral $\displaystyle\int \frac{dx}{2+\sqrt{6x-x^2-5}}$.

Solution: Observe that $6x - x^2 - 5 = 4 - (x-3)^2$. Put $x - 3 = 2\sin t$, $dx = 2\cos t\, dt$. Then

$$\int \frac{dx}{2+\sqrt{6x-x^2-5}} = \int \frac{\cos t}{1+\cos t}\, dt = \int dt - \int \frac{1}{1+\cos t}\, dt = t - \frac{1}{2}\int \sec^2 \frac{t}{2}\, dt = t - \tan\frac{t}{2} + C.$$

Now we have to convert this to a function of x. For this we will use the initial substitution $x - 3 = 2\sin t$ together with the following triangle.



However this triangle gives trigonometric functions of t, whereas we need to write $\tan(t/2)$. For this recall that

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \sin\beta\cos\alpha}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Dividing the numerator and denominator of the last fraction by $\cos\alpha\cos\beta$ we get

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.$$

Putting $\alpha = \beta = t/2$ and solving for $\tan(t/2)$ we get

$$\tan\frac{t}{2} = \pm\csc t - \cot t = \csc t - \cot t,$$

where the choice of the sign is dictated by the fact that $\tan(t/2)$ is defined at $t = 0$. Finally, using the above triangle we get

$$\int \frac{dx}{2+\sqrt{6x-x^2-5}} = \arcsin\left(\frac{x-3}{2}\right) + \frac{\sqrt{6x-x^2-5}-2}{x-3} + C.$$

Q-4) Find $T_4(f(x);0)$, the Taylor polynomial of order 4 at 0 of $f(x) = \int_0^x \tan(\sin t)\, dt$.

Solution: This is an exercise in taking derivatives!

$$\begin{aligned}
f(x) &= \int_0^x \tan(\sin t)\, dt, \\
f'(x) &= \tan(\sin x), \\
f''(x) &= \sec^2(\sin x)\cos x, \\
f'''(x) &= \sec^2(\sin x)[2\tan(\sin x)\cos^2 x - \sin x], \\
f^{(4)}(x) &= 2\sec^2(\sin x)\tan(\sin x)\cos x[2\tan(\sin x)\cos^2 x - \sin x] \\
&\quad + \sec^2(\sin x)[2\sec^2(\sin x)\cos^3 x - 4\tan(\sin x)\cos x \sin x - \cos x].
\end{aligned}$$

Using these we easily find that $f(0) = 0$, $f'(0) = 0$, $f''(0) = 1$, $f'''(0) = 0$, and $f^{(4)}(0) = 1$. This implies that

$$T_4(f(x);0) = \frac{1}{2}x^2 + \frac{1}{24}x^4.$$

Q-5) Evaluate the following limits:

a) $\displaystyle\lim_{t\to 0} \frac{6(\sin t - t) + t^3}{2(1-\cos t)\,t - t^3}$.

b) $\displaystyle\lim_{x\to 0} \frac{\arctan(\ln\cos x)}{x^2}$.

Solution-a) Here we use the Taylor expansions of sine and cosine functions to get

$$\lim_{t\to 0} \frac{6(\sin t - t) + t^3}{2(1-\cos t)\,t - t^3} = \lim_{t\to 0} \frac{\frac{1}{20}t^5 + o(t^5)}{-\frac{1}{12}t^5 + o(t^5)} = -\frac{3}{5}.$$

Solution-b) This problem requires an application of L'Hopital's rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{\arctan(\ln\cos x)}{x^2} &= \lim_{x\to 0} \frac{\dfrac{1}{1+(\ln\cos x)^2}\dfrac{1}{\cos x}(-\sin x)}{2x} \\
&= \lim_{x\to 0} \frac{\dfrac{1}{1+(\ln\cos x)^2}\dfrac{-1}{\cos x}}{2} \lim_{x\to 0}\frac{\sin x}{x} \\
&= -\frac{1}{2}.
\end{aligned}$$
