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Math 113 Calculus – Midterm Exam I
SOLUTIONS

Q-1) Prove by induction that $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$, for all integers $n \geq 1$.

Solution: For $n = 1$, both sides are 1. Assume for the induction hypothesis that $1^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$. Add $(n+1)^3 = n^3 + 3n^2 + 3n + 1$ to both sides of this equality and simplify the right hand side to obtain

$$\begin{aligned} 1^3 + \cdots + n^3 + (n+1)^3 &= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 \\ &= \frac{1}{4}(n^4 + 6n^3 + 13n^2 + 12n + 4) \\ &= \left(\frac{(n+1)(n+2)}{2}\right)^2. \end{aligned}$$

Thus starting from an assumption on n, we established the expected formula for $n+1$. This completes the induction argument and proves the formula for all $n \geq 1$.

Q-2) Calculate the following limits:

i) $\displaystyle\lim_{x\to 0} \frac{1 - \sqrt{1 - 3x^2 + 7x^3}}{x^2}$.

Solution i:

$$\begin{aligned} \frac{1 - \sqrt{1 - 3x^2 + 7x^3}}{x^2} &= \frac{1 - \sqrt{1 - 3x^2 + 7x^3}}{x^2} \cdot \frac{1 + \sqrt{1 - 3x^2 + 7x^3}}{1 + \sqrt{1 - 3x^2 + 7x^3}} \\ &= \frac{1 - (1 - 3x^2 + 7x^3)}{x^2(1 + \sqrt{1 - 3x^2 + 7x^3})} \\ &= \frac{3x^2 - 7x^3}{x^2(1 + \sqrt{1 - 3x^2 + 7x^3})} \\ &= \frac{3 - 7x}{1 + \sqrt{1 - 3x^2 + 7x^3}} \to \frac{3}{2} \quad \text{as} \quad x \to 0. \end{aligned}$$

ii) $\displaystyle\lim_{x\to 0} \frac{4\sin^2 x + 9 \sin x^2}{x^2}$.

Solution ii: $\displaystyle\frac{4\sin^2 x + 9\sin x^2}{x^2} = 4\left(\frac{\sin x}{x}\right)^2 + 9\left(\frac{\sin x^2}{x^2}\right) \to 4 \cdot 1^2 + 9 \cdot 1 = 13 \quad \text{as} \quad x \to 0.$

Q-3) Show that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

Solution: Assume that $\lim_{x\to 0}\sin\frac{1}{x} = A$ for some real number A. This means that for any $\epsilon > 0$ there is a corresponding $\delta > 0$ such that for all $x \in (-\delta, \delta)$ we have $|\sin\frac{1}{x} - A| < \epsilon$. Rewrite this last inequality as

$$-\epsilon < \sin\frac{1}{x} - A < \epsilon. \qquad (*)$$

Similarly for any other $y \in (-\delta, \delta)$ we have

$$-\epsilon < A - \sin\frac{1}{y} < \epsilon. \qquad (**)$$

Adding $(*)$ and $(**)$ side by side we find that for any $x, y \in (-\delta, \delta)$ we should have

$$-2\epsilon < \sin\frac{1}{x} - \sin\frac{1}{y} < 2\epsilon. \qquad (***)$$

Now set $x_n = \dfrac{1}{2n\pi + \pi/2}$ and $y_n = \dfrac{1}{2n\pi - \pi/2}$, where n is an integer. No matter how small δ is we can find a large integer n such that both x_n and y_n are in $(-\delta, \delta)$. Clearly $\sin\dfrac{1}{x_n} - \sin\dfrac{1}{y_n} = 2$, but this contradicts $(***)$ if we choose $0 < \epsilon \leq 1$.

This contradiction shows that our assumption of the existence of the above limit cannot hold. Hence $\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.

Q-4) A napkin-ring is obtained by drilling a cylindrical hole symmetrically through the center of a solid sphere. If the length of the hole is 4 units, find the volume of the napkin-ring.

Solution: Assume that the solid sphere is given by the equation $x^2 + y^2 + z^2 = r^2$, where r is its radius. With these coordinates assume that the cylindrical hole that is drilled out is expressed by the equation $x^2 + y^2 = c^2$ for some positive constant c. Now assume that we cut this napkin-ring by the yz-plane, or equivalently by the $x = 0$ plane. The resulting picture is depicted in the following figure. If however the napkin-ring is cut by a plane perpendicular to the yz-plane along the line AC, the slice obtained would look like a ring, consisting of a disk of radius AC out of which which a disk of radius AB is cut off. The area of this ring is $\pi(AC^2 - AB^2)$. So we set out to write this area explicitly:

Since the sphere $x^2 + y^2 + z^2 = r^2$ is cut off by the plane $x = 0$, the resulting circle of the figure has the form $y^2 + z^2 = r^2$.

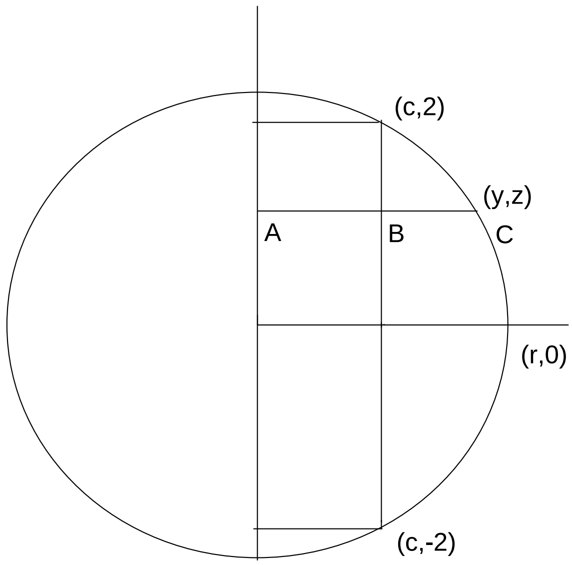
Since the point $(c, 2)$ is on this circle, we have $c^2 = r^2 - 4$. Observe that $c = AB$.

Since the point (y, z) is on the circle, we have as above $y^2 = r^2 - z^2$. Observe again that $y = AC$.

Thus the area of the slice is $\pi(AC^2 - AB^2) = \pi(4 - z^2)$.

To find the volume we have to add/integrate all these areas as z changes from -2 to 2.

$$\begin{aligned}\text{Volume} &= \int_{-2}^{2} \pi(4 - z^2)dz \\ &= \pi\left(4z - \frac{z^3}{3}\Big|_{-2}^{2}\right) = \frac{32}{3}\pi.\end{aligned}$$



The figure for the napkin-ring problem.

Q-5) Let f be a function such that $|f(u) - f(v)| \leq |u - v|$ for all u and v in an interval $[a, b]$.
i) Prove that f is continuous at each point of $[a, b]$.
ii) Assume that f is integrable on $[a, b]$. Prove that for any c in $[a, b]$, we have

$$\left| \int_a^b f(x)dx - (b-a)f(c) \right| \leq \frac{(b-a)^2}{2}.$$

Solution i: Let $c \in [a, b]$. Start with any $\epsilon > 0$. For any $x \in [a, b]$ we have $|f(x) - f(c)| \leq |x - c|$. Let $0 < \delta \leq \epsilon$ and set $U = (c - \delta, c + \delta) \cap [a, b]$ as the δ-neighbourhood of c in $[a, b]$. Then for all $x \in U$, we have $|f(x) - f(c)| \leq |x - c| < \delta \leq \epsilon$, which establishes the continuity of f at c.

Solution ii:

$$\begin{aligned}
\left| \int_a^b f(x)dx - (b-a)f(c) \right| &= \left| \int_a^b (f(x) - f(c))dx \right| \\
&\leq \int_a^b |f(x) - f(c)|\, dx \\
&\leq \int_a^b |x - c|\, dx \\
&= \int_a^c (c - x)\, dx + \int_c^b (x - c)\, dx \\
&= \left(cx - \frac{1}{2}x^2 \Big|_a^c \right) + \left(\frac{1}{2}x^2 - cx \Big|_c^b \right) \\
&= \frac{1}{2}(b-a)^2 + (a-c)(b-c) \\
&\leq \frac{1}{2}(b-a)^2, \text{ since } (a-c)(b-c) \leq 0.
\end{aligned}$$