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Math 113 Calculus – Midterm Exam II
SOLUTIONS

Q-1) Find the minimum and maximum values, if they exist, of the function

$$f(x) = x^{\log x}, \quad 0 < x < \infty.$$

Solution: Clearly $f(x)$ is unbounded as $x \to \infty$, so no maximum exists. We can only expect to find a minimum value. For this we examine the derivative of the function. Note that $f(x) = x^{\log x} = e^{(\log x)^2}$, so $f'(x) = e^{(\log x)^2}\ (2\log x)\left(\frac{1}{x}\right)$.

$f'(x) = 0$ for $x = 1$. Examining the sign of the derivative we find:
$f'(x) < 0$ for $0 < x < 1$, and $f'(x) > 0$ for $1 < x < \infty$.
So $f(x)$ descend to $f(1) = 1$ and from there on increases to infinity. Hence the minimum value is 1.

Q-2) Evaluate $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^2}$.

Solution: We start with the integral $\displaystyle\int \frac{dx}{1+x^2}$ and apply integration by parts with $du = \frac{1}{1+x^2}$ to obtain

$$\int \frac{dx}{1+x^2} = \frac{x}{1+x^2} + 2\int \frac{x^2}{(1+x^2)^2}\,dx.$$

We observe that

$$\int \frac{x^2}{(1+x^2)^2}\,dx = \int \frac{x^2+1-1}{(1+x^2)^2}\,dx = \int \frac{dx}{1+x^2} - \int \frac{dx}{(1+x^2)^2}.$$

We already know that $\displaystyle\int \frac{dx}{1+x^2} = \arctan x + C$. Putting these together we find

$$\int \frac{dx}{(1+x^2)^2} = \frac{1}{2}\left(\frac{x}{1+x^2} + \arctan x\right) + C$$

and finally

$$\int_0^1 \frac{dx}{(1+x^2)^2} = \frac{2+\pi}{8}.$$

Q-3) Evaluate $\displaystyle\int \frac{x+2}{x^2+x+1}\,dx$.

Solution: We first observe that

$$\frac{x+2}{x^2+x+1} = \frac{1}{2}\,\frac{2x+1}{x^2+x+1} + \frac{3}{2}\,\frac{1}{x^2+x+1}.$$

Using substitution in the following integral with $u = x^2 + x + 1$ we find

$$\int \frac{2x+1}{x^2+x+1}\,dx = \int \frac{du}{u} = \log u + C = \log(x^2+x+1) + C.$$

And for the other fraction, after completing to a square we have

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + C.$$

Putting these together we get

$$\int \frac{x+2}{x^2+x+1}\,dx = \frac{1}{2}\log(x^2+x+1) + \sqrt{3}\arctan\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) + C.$$

Q-4) Among all circular right cones inscribed in a sphere of radius 6 units, find the volume of the one

a) with the minimum volume.

b) with the maximum volume.

Solution: If the radius of the cone is r and its height is h, then the volume is $V = \frac{\pi}{3}r^2h$.
Let x measure how far below the center of the sphere lies the base of the cone.
Then $r^2 = 6^2 - x^2$, $h = x + 6$ and the volume becomes

$$V(x) = \frac{\pi}{3}(36 - x^2)(6+x), \quad -6 \le x \le 6.$$

We find that $V'(x) = 0$ for $x = -6$ and for $x = 2$. Calculating $V(x)$ at the critical and the end points gives

$$V(-6) = 0, \quad V(2) = \frac{256}{3}\pi, \quad V(6) = 0.$$

This then answers both questions about the minimum and maximum values.

Q-5) The graph of a continuous function f is revolved around the x-axis to obtain a solid of revolution. For every $a \in [0, \pi/2]$, the volume of the solid obtained by revolving the graph of $y = f(x)$ for $x \in [0, a]$ is given by

$$2^a + \arctan a + \cosh a + C$$

for some constant C. For which value of C does such a function f exist? Find f for that value of C.

Solution: For the existence of such a function we must have $(2^a + \arctan a + \cosh a + C)\,|_{a=0} = 0$, which corresponds to saying that if we have no graph, then we have no volume. This gives $C = -2$. The volume of the solid of revolution formula gives

$$\pi \int_0^x f^2(t)\,dt = 2^x + \arctan x + \cosh x - 2, \quad x \in (0, \pi/2).$$

Taking the derivative of both sides with respect to x, and using the fundamental theorem of calculus on the left hand side gives

$$\pi f^2(x) = 2^x \log 2 + \frac{1}{1+x^2} + \sinh x,$$

so

$$f(x) = \sqrt{\frac{1}{\pi}\left(2^{\,x}\log 2 + \frac{1}{1+x^2} + \sinh x\right)}.$$