Exercise 12, page 45 of Apostol’s Calculus:

(a) Use the binomial theorem to prove that for a positive integer we have

\[
\left( 1 + \frac{1}{n} \right)^n = 1 + \sum_{k=1}^{n} \left\{ \frac{1}{k!} \prod_{r=0}^{k-1} \left( 1 - \frac{r}{n} \right) \right\}.
\]

(b) If \( n > 1 \), use part (a) and the fact that \( 2^n < n! \) for all \( n \geq 4 \), to deduce the inequalities

\[
2 < \left( 1 + \frac{1}{n} \right)^n < 1 + \sum_{k=1}^{n} \frac{1}{k!} < 3.
\]

Solution:

(a) Binomial theorem says \((a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k\), where \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \) is the binomial coefficient. Using this we write

\[
\left( 1 + \frac{1}{n} \right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = 1 + \sum_{k=1}^{n} \binom{n}{k} \frac{1}{n^k} = 1 + \sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} = 1 + \sum_{k=1}^{n} \frac{1}{k!} \left\{ \frac{n!}{(n-k)!} \frac{1}{n^k} \right\} = 1 + \sum_{k=1}^{n} \frac{1}{k!} \left\{ \prod_{r=0}^{k-1} \left( \frac{n-r}{n} \right) \right\} = 1 + \sum_{k=1}^{n} \frac{1}{k!} \left\{ \prod_{r=0}^{k-1} \left( 1 - \frac{r}{n} \right) \right\}.
\]

(b) First recall that \( 2^n < n! \) for \( n \geq 4 \), which can be easily proven by induction. We will use this in the form \( \frac{1}{n!} < \frac{1}{2^n} \) for \( n \geq 4 \).

Now back to our problem. Clearly each \( 1 - \frac{r}{n} < 1 \), so \( \prod_{r=0}^{k-1} \left( 1 - \frac{r}{n} \right) < 1 \). Hence from the first part of this solution we get

\[
\left( 1 + \frac{1}{n} \right)^n < 1 + \sum_{k=1}^{n} \frac{1}{k!}.
\]

For the second inequality we simply add the terms on the right hand side. By direct computation
we see that the right hand side is $< 3$ for $n = 2, 3$. So take $n \geq 4$.

\[
1 + \sum_{k=1}^{n} \frac{1}{k!} = 1 + \frac{1}{2!} + \frac{1}{3!} + \left(\frac{1}{4!} + \cdots + \frac{1}{n!}\right)
\]

\[
= 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!}
\]

\[
< 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n}
\]

\[
= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n} - \frac{1}{n!}
\]

\[
< 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n}
\]

\[
= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2^n}
\]

This proves the inequalities

\[
\left(1 + \frac{1}{n}\right)^n < 1 + \sum_{k=1}^{n} \frac{1}{k!} < 3.
\]

For the remaining inequality first observe that for $n = 2$, we clearly have $2 < (1 + 1/2)^2 = 9/4$. For $n > 2$ we use the result of part (a):

\[
\left(1 + \frac{1}{n}\right)^n = 1 + \sum_{k=1}^{n} \frac{1}{k!} \left\{ \prod_{r=0}^{k-1} \left(1 - \frac{r}{n}\right) \right\}
\]

\[
= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \left\{ \prod_{r=0}^{k-1} \left(1 - \frac{r}{n}\right) \right\}
\]

\[
> 2, \text{ since each term in the summation is positive.}
\]

Hence we finally get, for all $n > 1$,

\[
2 < \left(1 + \frac{1}{n}\right)^n < 1 + \sum_{k=1}^{n} \frac{1}{k!} < 3.
\]

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