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Exercise 12, page 45 of Apostol's Calculus:

(a) Use the binomial theorem to prove that for n a positive integer we have

$$\left(1+\frac{1}{n}\right)^n = 1+\sum_{k=1}^{n}\left\{\frac{1}{k!}\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}.$$

(b) If $n>1$, use part (a) and the fact that $2^n < n!$ for all $n \geq 4$, to deduce the inequalities

$$2 < \left(1+\frac{1}{n}\right)^n < 1+\sum_{k=1}^{n}\frac{1}{k!} < 3.$$

Solution:

(a) Binomial theorem says $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. Using this we write

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\
&= 1+\sum_{k=1}^{n}\binom{n}{k}\frac{1}{n^k} \\
&= 1+\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}\frac{1}{n^k} \\
&= 1+\sum_{k=1}^{n}\frac{1}{k!}\left\{\frac{n!}{(n-k)!}\frac{1}{n^k}\right\} \\
&= 1+\sum_{k=1}^{n}\frac{1}{k!}\left\{\frac{(n-k+1)(n-k+2)\cdots(n-1)(n)}{n^k}\right\} \\
&= 1+\sum_{k=1}^{n}\frac{1}{k!}\left\{\prod_{r=0}^{k-1}\left(\frac{(n-r)}{n}\right)\right\} \\
&= 1+\sum_{k=1}^{n}\frac{1}{k!}\left\{\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\}.
\end{aligned}$$

(b) First recall that $2^n < n!$ for $n \geq 4$, which can be easily proven by induction. We will use this in the form $\frac{1}{n!} < \frac{1}{2^n}$ for $n \geq 4$.

Now back to our problem. Clearly each $1-\frac{r}{n} < 1$, so $\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right) < 1$. Hence from the first part of this solution we get

$$\left(1+\frac{1}{n}\right)^n < 1+\sum_{k=1}^{n}\frac{1}{k!}.$$

For the second inequality we simply add the terms on the right hand side. By direct computation we see that the right hand side is < 3 for $n = 2, 3$. So take $n \geq 4$.

$$\begin{aligned}
1+\sum_{k=1}^{n}\frac{1}{k!} &= 1+\frac{1}{2!}+\frac{1}{3!}+\left(\frac{1}{4!}+\cdots+\frac{1}{n!}\right) \\
&= \frac{8}{3}+\left(\frac{1}{4!}+\cdots+\frac{1}{n!}\right) \\
&< \frac{8}{3}+\left(\frac{1}{2^4}+\cdots+\frac{1}{2^n}\right) \\
&= \frac{8}{3}+\frac{1}{2^4}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{n-4}}\right) \\
&= \frac{8}{3}+\frac{1}{16}\left(\frac{1-(1/2)^{n-3}}{1-(1/2)}\right) \\
&= \frac{8}{3}+\frac{1}{8}\left(1-2^{3-n}\right) \\
&< \frac{8}{3}+\frac{1}{8}=\frac{67}{24}<3.
\end{aligned}$$

This proves the inequalities

$$\left(1+\frac{1}{n}\right)^n < 1+\sum_{k=1}^{n}\frac{1}{k!} < 3.$$

For the remaining inequality first observe that for $n = 2$, we clearly have $2 < (1 + 1/2)^2 = 9/4$. For $n > 2$ we use the result of part (a):

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= 1+\sum_{k=1}^{n}\frac{1}{k!}\left\{\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\} \\
&= 1+1+\sum_{k=2}^{n}\frac{1}{k!}\left\{\prod_{r=0}^{k-1}\left(1-\frac{r}{n}\right)\right\} \\
&> 2, \quad \text{since each term in the summation is positive.}
\end{aligned}$$

Hence we finally get, for all $n > 1$,

$$2 < \left(1+\frac{1}{n}\right)^n < 1+\sum_{k=1}^{n}\frac{1}{k!} < 3.$$

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