Math 113 Homework 2
Due: 9 November 2004 Tuesday class hour.
Instructor: Ali Sinan Sertöz

Q-1) Exercise 3 on page 155. Use the identity $1+x^6 = (1+x^2)(1-x^2+x^4)$ and the weighted mean value theorem for integrals to prove that for $a > 0$, we have

$$\frac{1}{1+a^2}\left(a - \frac{a^3}{3} + \frac{a^5}{5}\right) \leq \int_0^a \frac{dx}{1+x^2} \leq a - \frac{a^3}{3} + \frac{a^5}{5}.$$

Solution: The integrand, $\frac{1}{1+x^2}$ can be written as $f(x)g(x)$ where $f(x) = \frac{1}{1+x^6}$ and $g(x) = 1 - x^2 + x^4$. We observe that g does not change sign on $[0,a]$. Moreover,

$\displaystyle\int_0^a g(x)\,dx = a - \frac{a^3}{3} + \frac{a^5}{5}$, min f on $[0,a]$ is $\displaystyle\frac{1}{1+a^2}$, and max f on $[0,a]$ is 1.

The weighted mean value theorem in this case says that $\displaystyle\int_0^a f(x)g(x)\,dx = f(c)\int_0^a g(x)\,dx = f(c)\left(a - \frac{a^3}{3} + \frac{a^5}{5}\right)$ for some $c \in [0,a]$. Since no matter where c is in $[0,a]$ we must have $\displaystyle\frac{1}{1+a^2} \leq f(c) \leq 1$, the required result follows.

Q-2) Exercise 4 on page 155. One of the following two statements is incorrect. Explain why it is wrong.

(a) The integral $\int_{2\pi}^{4\pi} (\sin t)/t\,dt > 0$ because $\int_{2\pi}^{4\pi} (\sin t)/t\,dt > \int_{3\pi}^{4\pi} |\sin t|/t\,dt$.

(b) The integral $\int_{2\pi}^{4\pi} (\sin t)/t\,dt = 0$ because, by the weighted mean value theorem for integrals, for some c between 2π and 4π we have

$$\int_{2\pi}^{4\pi} \frac{\sin t}{t}\,dt = \frac{1}{c}\int_{2\pi}^{4\pi} \sin t\,dt = \frac{\cos(2\pi) - \cos(4\pi)}{c} = 0.$$

Solution: This is an exercise in reading and understanding the statements of theorems. The weighted mean value theorem works only when the function g does not change sign in the given interval. If you follow the proof of that theorem you will see that this fact is crucially used. Since $\sin t$ changes sign in the interval $[2\pi, 4\pi]$, this theorem cannot be used here. Therefore statement (b) is incorrect.

Q-3) Let $f : [0, 1] \to [0, 1]$ be a 2-1 onto function. This means that for every $y \in [0, 1]$ there are exactly two points x_1 and x_2 in $[0, 1]$ such that $f(x_1) = f(x_2) = y$.

a) Show that f is not continuous on $[0, 1]$.

b) Construct such an f.

Solution:

(a): Suppose f is continuous. Let $x_1 < x_2 \in [0, 1]$ be the two points where $f(x_1) = f(x_2) = 1$. Let $c_0 \in (x_1, x_2)$ and let k be any value with $f(c_0) < k < 1$.

Since $f : [x_1, c_0] \longrightarrow [0, 1]$ is continuous, there is a point $c_1 \in (x_1, c_0)$ such that $f(c_1) = k$.

Similarly since $f : [c_0, x_2] \longrightarrow [0, 1]$ is continuous, there is a point $c_2 \in (c_0, x_2)$ such that $f(c_2) = k$.

Thus all the values in $(f(c_0), 1]$ are already taken twice by f in the interval $[x_1, x_2]$. In particular the value k is already taken twice here.

Case 1: $0 < x_1$. Since f is two-to-one, and since all the values in $(f(c_0), 1]$ are already taken twice by f in the interval $[x_1, x_2]$, we must have $f(0) \leq f(c_0)$. On the other hand, $f : [0, x_1] \longrightarrow [0, 1]$ is continuous and we have $f(0) \leq f(c_0) < k < f(x_1) = 1$. Therefore by the intermediate value theorem for continuous functions, there exists a point $x_3 \in (0, x_1)$ with $f(x_3) = k$. But this is the third time f is attaining the value k, and this contradicts the two-to-one property of f.

Case 2: $0 = x_1 < x_2 < 1$. Repeat the above argument by considering f on $[x_2, 1]$.

Case 3: $0 = x_1 < x_2 = 1$. Now let $t_1 < t_2 \in (0, 1)$ be the points where $f(t_1) = f(t_2) = 0$. Since $f(0) = 1$ and $f(t_1) = 0$, f takes all the values in $[0, 1]$ at least once in the interval $[0, t_1]$. Similarly, since $f(t_2) = 0$ and $f(1) = 1$, f again takes all the values in $[0, 1]$ at least once in the interval $[t_2, 1]$. Thus no value is left for the function on the interval (t_1, t_2), which contradicts the fact that f is defined there.

So f cannot be continuous.

(b): There are many ways to construct such functions. In all cases you use the fact that there are infinitely many points in $[0, 1]$. Here is one such function.

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \in [\frac{1}{2}, 1]. \\ 4x & \text{if } x = \dfrac{1}{2^n} \text{ for some integer } n > 1. \\ 2x & \text{Otherwise.} \end{cases}$$

---

comments to sertoz@bilkent.edu.tr