Date: 7 January 2005, Friday
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Math 113 Calculus – Makeup Exam – Solutions

Q-1) Write the derivatives of the following functions with respect to x. Do not simplify your answers. No partial credits!

a) $f(x) = e^{\cos x} + (\cos x)^{\cos x}$, $f'(x) = e^{\cos x}(-\sin x) + (\cos x)^{\cos x}\left(-\sin x \ln\cos x + \cos x \dfrac{-\sin x}{\cos x}\right)$.

b) $f(x) = x^{\ln x}$, $f'(x) = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$.

c) $f(x) = x^{\pi^x}$, $f'(x) = x^{\pi^x}\left(\dfrac{\pi^x}{x} + \pi^x \ \ln\pi \ln x\right)$.

d) $f(x) = \cosh^{\sinh x} x + 2^{\pi}$, $f'(x) = \cosh^{\sinh x} x \left(\cosh x \ln\cosh x + \sinh x \dfrac{\sinh x}{\cosh x}\right)$.

Q-2) Calculate the following derivatives. Write the answers inside the given boxes. No partial credits!

a) If $x^3 + x^2y + xy^3 + y^4 + 17 = 0$, and $x = -3$, $y = 2$, then $y' = ?$

Implicitly differentiating the equation we get $3x^2 + 2xy + x^2y' + y^3 + 3xy^2y' + 4y^3y' = 0$. Putting in $x = -3$ and $y = 2$ we get $23 + 5y' = 0$ from which we get $y' = \boxed{-\dfrac{\mathbf{23}}{\mathbf{5}}}$.

b) $f(x) = \dfrac{x^3 + x}{x^7}$, $\quad x(t) = \sec t$. $\left.\dfrac{df}{dt}\right|_{t=\pi/4} = ?$

$f'(x) = \dfrac{(3x^2+1)(x^7) - (x^3+x)(7x^6)}{x^{14}}$, $x(\pi/4) = \sqrt{2}$, $\mathbf{f'(\sqrt{2}) = -\dfrac{7\sqrt{2}}{8}}$. $x'(t) = \sec t \tan t$, $\mathbf{x'(\pi/4) = \sqrt{2}}$.

$$\left.\frac{df}{dt}\right|_{t=\pi/4} = \left.\frac{df}{dx}\right|_{x=\sqrt{2}} \left.\frac{dx}{dt}\right|_{t=\pi/4} = -\frac{\mathbf{7\sqrt{2}}}{\mathbf{8}} \cdot \mathbf{\sqrt{2}} = \boxed{-\frac{\mathbf{7}}{\mathbf{4}}}$$

c) $f(x) = \tan^3(\pi x) + 3^{\tan(\pi x)}$, $\quad x(t) = \dfrac{t^3 - 1}{t^2 + 1}$. $\left.\dfrac{df}{dt}\right|_{t=1} = ?$

$f'(x) = (3\tan^2 \pi x)(\pi \sec^2 \pi x) + 3^{\tan(\pi x)}(\pi \sec^2 \pi x \ \ln 3)$, $x(1) = 0$, $\mathbf{f'(0) = \pi \ln 3}$.

$$x'(t) = \frac{3t^2(t^2+1)-(t^3-1)(2t)}{(t^2+1)^2}, \ \mathbf{x'(1)} = \frac{\mathbf{3}}{\mathbf{2}}.$$

$$\left.\frac{df}{dt}\right|_{t=1} = \left.\frac{df}{dx}\right|_{x=1/4} \left.\frac{dx}{dt}\right|_{t=1} = \pi \ln \mathbf{3} \cdot \frac{3}{2} = \boxed{\frac{\mathbf{3}\pi \ln \mathbf{3}}{\mathbf{2}}}$$

d) $f(x) = x^x, \quad x(t) = 2^{\,\ln t}. \ \left.\frac{df}{dt}\right|_{t=1} = ?$

$f'(x) = x^x(\ln x + 1)$, $x(1) = 1$, $\mathbf{f'(1) = 1}$. $x'(t) = 2^{\,\ln t}\left(\frac{\ln 2}{t}\right)$, $\mathbf{x'(1)} = \ln \mathbf{2}$.

$$\left.\frac{df}{dt}\right|_{t=1} = \left.\frac{df}{dx}\right|_{x=1} \left.\frac{dx}{dt}\right|_{t=1} = \mathbf{1} \cdot \ln \mathbf{2} = \boxed{\ln \mathbf{2}}$$

Q-3) Find the minimum and the maximum values of the function $f(x) = \frac{x}{(x-2)^2} + 1$ on $(-\infty, 0]$.

$f'(x) = -\frac{x+2}{(x-2)^3} = 0$ when $x = -2$.

$f''(x) = \frac{2(x+4)}{(x-2)^4}$ and $f''(-2) > 0$, so $x = -2$ is a local minimum point.

Now we check the end points and the critical point:

$\lim_{x\to-\infty} f(x) = 1$ and since $\frac{x}{(x-2)^2} < 0$ for $x < 0$, we must have $f(x) < 1$ for all $x < 0$.

$f(-2) = \frac{7}{8}$.

$f(0) = 1$.

Therefore the maximum value is 1, and the minimum value is $\frac{7}{8}$.

Q-4) Evaluate the integral $\int x \sec x \tan x \, dx$.

Use integration by parts method with $u = x$ and $dv = \sec x \tan x \, dx$. Then $du = dx$ and $v = \sec x$. This gives

$$\int x \sec x \tan x \, dx = x \sec x - \int \sec x \, dx = x \sec x - \ln|\sec x + \tan x| + C.$$

Q-5) Find the Taylor polynomial of degree 5 of $\ln(1+x)$ at $x = 0$.

Hint: You might want to start with $\frac{1}{1-t} = 1 + t + t^2 + \cdots + t^n + \frac{t^{n+1}}{1-t}$.

Put $t = -x$ and integrate from 0 to x to obtain

$$T_5(\ln(1+x)) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5.$$