Math 113 Calculus – Midterm Exam I – Solutions

Q-1) Let $M = \{ x \in \mathbb{R} \mid x < \sqrt{5} \}$. Prove that $\sqrt{5}$ is the supremum of $M$. Moreover show that for any $\epsilon > 0$, there exists at least one element $y \in M$ such that $\sqrt{5} - \epsilon < y$.

Solution: Assume that $\sqrt{5}$ is not the supremum of $M$. On the other hand, $\sqrt{5}$ is an upper bound for $M$, and we know that being nonempty and bounded from above, $M$ has supremum. Let $s$ be the supremum of $M$. Then $s < \sqrt{5}$. Let $t = (\sqrt{5} + s)/2$. But $s < t < \sqrt{5}$ gives us two contradictory results: $t \in M$ and $t > \sup M$. We reached this contradiction by starting with the assumption that $\sqrt{5}$ is not the supremum of $M$. Therefore that assumption must be wrong, and indeed $\sqrt{5} = \sup M$.

The other claim is extremely easy to prove: let $y = \sqrt{5} - (\epsilon/2)$.

Q-2-a) Prove by induction that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all integers $n \geq 1$.

Q-2-b) Prove by induction one of the following statements:

(i) $4 + 13 + 28 + \cdots + (3n^2 + 1) \leq n^3 + 3n$, for all integers $n \geq 1$.
(ii) $4 + 13 + 28 + \cdots + (3n^2 + 1) = n^3 + 3n$, for all integers $n \geq 1$.
(iii) $4 + 13 + 28 + \cdots + (3n^2 + 1) \geq n^3 + 3n$, for all integers $n \geq 1$.

Solution: Q-2-a) The statement is true for $n = 1$. Assume that it is true for some $n$, and $2(n+1) - 1$ to both sides of the equality

\[
1 + 3 + 5 + \cdots + (2n - 1) = n^2 \\
2(n+1) - 1 = 2n + 1
\]

adding up side by side, we get:

\[
1 + 3 + 5 + \cdots + (2(n+1) - 1) = (n+1)^2
\]

which shows that the statement is also true for $n+1$ when it is true for $n$. This completes the induction argument and proves the claim for all $n \geq 1$.

Solution: Q-2-b) All three statements are true for $n = 1$, but only the last one is true for $n = 2$. Therefore we try to prove (iii). We already know that it is true for $n = 1$. We assume that it is true for some $n$. We add $3(n+1)^2 + 1$ to both sides of the inequality

\[
4 + 13 + 28 + \cdots + (3n^2 + 1) \geq n^3 + 3n \\
3(n+1)^2 + 1 = 3n^2 + 6n + 4
\]

adding up side by side, we get:

\[
4 + 13 + 28 + \cdots + (3(n+1)^2 + 1) \geq (n+1)^3 + 3(n+1) + 3n \\
\geq (n+1)^3 + 3(n+1)
\]
Q-3) Define a function $f : [0, 1] \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Is $f$ integrable on $[0, 1]$? If yes, calculate $\int_0^1 f(x)\,dx$. If not, then explain why.

Solution: Let as usual $S$ be the integrals of all nonnegative step functions $s$ on $[0, 1]$ with $s(x) \leq f(x)$. There is only $s = 0$ step function satisfying this condition, so $S = \{0\}$. Hence sup $S = 0$.

Let $T$ be the set of integrals of all step functions $t$ on $[0, 1]$ such that $f(x) \leq t(x)$.

Consider the step function $h$ defined as $h(x) = 0$ for $0 \leq x < 1/2$, and $h(x) = 1/4$ for $1/2 \leq x \leq 1$. Then for all $x \in [0, 1]$ we have $0 \leq h(x) \leq f(x) \leq t(x)$ for every step function $t \geq f$ on $[0, 1]$. In particular $1/8 = \int_0^1 h(x)\,dx \leq \int_0^1 t(x)\,dx$. Therefore inf $T \geq 1/8 > 0 = sup S$, and the integral of $f$ does not exist.

Q-4) Calculate the area bounded between the curve $f(x) = x^3 - 4x$ and the $x$-axis, from $x = -1$ to $x = 1$.

Solution: Note that $f(-x) = -f(x)$ and $f(x) > 0$ for $x < 0$. Then the required area is

$$\text{Area} = 2 \int_{-1}^0 (x^3 - 4x)\,dx$$

$$= 2 \left( \frac{x^4}{4} - 2x^2 \right|_{-1}^0$$

$$= \frac{7}{2}.$$

Q-5) The line $y = x/5$ intersects the graph of $y = \sin x$ at $x = 0$ and $x = \alpha = 2.595739080$ when $x \geq 0$. Let $R$ denote the region that they thus bound. Set up an integral which calculate the volume of the solid obtained by revolving the region $R$ around

(i) $x$-axis.
(ii) $y$-axis.

Do not evaluate the integrals. (we will be able to evaluate these integrals in chapter 5.)

Solution: (i) $\pi \int_0^\alpha \left( \sin^2 x - \frac{x^2}{25} \right)\,dx$.

Solution: (ii) $2\pi \int_0^\alpha x \left( \sin x - \frac{x}{5} \right)\,dx$.  

which completes the proof, as above.