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Math 113 Calculus – Midterm Exam I – Solutions

Q-1) Let $M = \{x \in \mathbb{R} \mid x < \sqrt{5}\ \}$. Prove that $\sqrt{5}$ is the supremum of M. Moreover show that for any $\epsilon > 0$, there exists at least one element $y \in M$ such that $\sqrt{5} - \epsilon < y$.

Solution: Assume that $\sqrt{5}$ is not the supremum of M. On the other hand, $\sqrt{5}$ is an upper bound for M, and we know that being nonempty and bounded from above, M has supremum. Let s be the supremum of M. Then $s < \sqrt{5}$. Let $t = (\sqrt{5} + s)/2$. But $s < t < \sqrt{5}$ gives us two contradictory results: $t \in M$ and $t > \sup M$. We reached this contradiction by starting with the assumption that $\sqrt{5}$ is not the supremum of M. Therefore that assumption must be wrong, and indeed $\sqrt{5} = \sup M$.

The other claim is extremely easy to prove: let $y = \sqrt{5} - (\epsilon/2)$.

Q-2-a) Prove by induction that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all integers $n \geq 1$.

Q-2-b) Prove by induction one of the following statements:

(i) $4 + 13 + 28 + \cdots + (3n^2 + 1) \leq n^3 + 3n$, for all integers $n \geq 1$.

(ii) $4 + 13 + 28 + \cdots + (3n^2 + 1) = n^3 + 3n$, for all integers $n \geq 1$.

(iii) $4 + 13 + 28 + \cdots + (3n^2 + 1) \geq n^3 + 3n$, for all integers $n \geq 1$.

Solution: Q-2-a) The statement is true for $n = 1$. Assume that it is true for some n, and $2(n + 1) - 1$ to both sides of the equality

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2n - 1) &= n^2 \\ 2(n + 1) - 1 &= 2n + 1 \\ \text{adding up side by side, we get:} & \\ 1 + 3 + 5 + \cdots + (2(n + 1) - 1) &= (n + 1)^2 \end{aligned}$$

which shows that the statement is also true for $n + 1$ when it is true for n. This completes the induction argument and proves the claim for all $n \geq 1$.

Solution: Q-2-b) All three statements are true for $n = 1$, but only the last one is true for $n = 2$. Therefore we try to prove (iii). We already know that it is true for $n = 1$. We assume that it is true for some n. We add $3(n + 1)^2 + 1$ to both sides of the inequality

$$\begin{aligned} 4 + 13 + 28 + \cdots + (3n^2 + 1) &\geq n^3 + 3n \\ 3(n + 1)^2 + 1 &= 3n^2 + 6n + 4 \\ \text{adding up side by side, we get:} & \\ 4 + 13 + 28 + \cdots + (3(n + 1)^2 + 1) &\geq (n + 1)^3 + 3(n + 1) + 3n \\ &\geq (n + 1)^3 + 3(n + 1) \end{aligned}$$

which completes the proof, as above.

Q-3) Define a function $f : [0,1] \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Is f integrable on $[0,1]$? If *yes*, calculate $\int_0^1 f(x)dx$. If *not*, then explain why.

Solution: Let as usual S be the integrals of all nonnegative step functions s on $[0,1]$ with $s(x) \leq f(x)$. There is only $s = 0$ step function satisfying this condition, so $S = \{0\}$. Hence $\sup S = 0$.

Let T be the set of integrals of all step functions t on $[0,1]$ such that $f(x) \leq t(x)$.

Consider the step function h defined as $h(x) = 0$ for $0 \leq x < 1/2$, and $h(x) = 1/4$ for $1/2 \leq x \leq 1$. Then for all $x \in [0,1]$ we have $0 \leq h(x) \leq f(x) \leq t(x)$ for every step function $t \geq f$ on $[0,1]$. In particular $1/8 = \int_0^1 h(x)dx \leq \int_0^1 t(x)dx$. Therefore $\inf T \geq 1/8 > 0 = \sup S$, and the integral of f does not exist.

Q-4) Calculate the area bounded between the curve $f(x) = x^3 - 4x$ and the x-axis, from $x = -1$ to $x = 1$.

Solution: Note that $f(-x) = -f(x)$ and $f(x) > 0$ for $x < 0$. Then the required area is

$$\begin{aligned} \text{Area} &= 2\int_{-1}^{0} (x^3 - 4x)dx \\ &= 2\left(\frac{x^4}{4} - 2x^2\Big|_{-1}^{0}\right) \\ &= \frac{7}{2}. \end{aligned}$$

Q-5) The line $y = x/5$ intersects the graph of $y = \sin x$ at $x = 0$ and $x = \alpha = 2.595739080$ when $x \geq 0$. Let R denote the region that they thus bound. Set up an integral which calculate the volume of the solid obtained by revolving the region R around
(i) x-axis.
(ii) y-axis.
Do <u>not</u> evaluate the integrals. (we will be able to evaluate these integrals in chapter 5.)

Solution: (i) $\pi \displaystyle\int_0^{\alpha} \left(\sin^2(x) - \frac{x^2}{25}\right) dx.$

Solution: (ii) $2\pi \displaystyle\int_0^{\alpha} x\left(\sin(x) - \frac{x}{5}\right) dx.$
