Date: 20 November 2004, Saturday
Instructor: Ali Sinan Sertöz
Time: 10:00-12:00

Math 113 Calculus – Midterm Exam II – Solutions

Q-1) Write the derivatives of the given functions. Do not simplify your answer. No partial credits!

a) $f(x) = x\tan x + \sqrt{1+x^2}$, $f'(x) = \tan x + x\sec^2 x + \dfrac{x}{\sqrt{1+x^2}}$

b) $f(x) = \dfrac{x^3+3x-1}{x^4+x+1}$, $f'(x) = \dfrac{(3x^2+3)(x^4+x+1)-(x^3+3x-1)(4x^3+1)}{(x^4+x+1)^2}$

c) $f(x) = 1+\sqrt{2+\sqrt{3+x^2}}$, $f'(x) = \dfrac{1}{2\sqrt{2+\sqrt{3+x^2}}} \cdot \dfrac{x}{\sqrt{3+x^2}}$

d) $f(x) = \left(1+3x^5\right)(x+\cos x)\left(\dfrac{\sin x}{1+x}\right)$,

$$f'(x) = (15x^4)(x+\cos x)\left(\tfrac{\sin x}{1+x}\right) + (1+3x^5)(1-\sin x)\left(\tfrac{\sin x}{1+x}\right) + (1+3x^5)(x+\cos x)\left(\tfrac{(\cos x)(1+x)-\sin x}{(1+x)^2}\right)$$

Q-2) Calculate the following derivatives. No partial credits!

a) $f(x) = x\cos x$, $x(t) = (1+t)/(1-t)$, $\left.\dfrac{df}{dt}\right|_{t=-1} = \boxed{\mathbf{\dfrac{1}{2}}}$

$$x'(t) = \frac{2}{(1-t)^2}, \quad x'(-1) = \boxed{\frac{1}{2}}, \quad x(-1) = 0.$$

$$f'(x) = \cos x - x\sin x, \quad f'(0) = \boxed{1},$$

$$\left.\frac{df}{dt}\right|_{t=-1} = f'(0)\cdot x'(-1) = \mathbf{\frac{1}{2}}$$

b) $f(x) = x^2+x+1$, $g(x) = \cos x$, $h(x) = \dfrac{x+\pi}{x^3+3}$, $(f\circ g\circ h)'(0) = \boxed{-\mathbf{\dfrac{1}{\sqrt{3}}}}$

$$h'(x) = \frac{x^3+3-(x+\pi)(3x^2)}{(x^3+3)^2}, \quad h'(0) = \boxed{\frac{1}{3}}, \quad h(0) = \tfrac{\pi}{3}.$$

$g'(x) = -\sin x, \quad g'(\frac{\pi}{3}) = \boxed{-\dfrac{\sqrt{3}}{2}}, \quad g(\frac{\pi}{3}) = \frac{1}{2}$

$f'(x) = 2x + 1, \quad f'(\frac{1}{2}) = \boxed{2}$

$(f \circ g \circ h)'(0) = f'(\frac{1}{2}) \cdot g'(\frac{\pi}{3}) \cdot h'(0) = -\frac{\mathbf{1}}{\sqrt{\mathbf{3}}}$

c) If $x^3 + x^2y + xy^3 + y^4 + 17 = 0$, and $x = -3$, $y = 2$, then $y' = \boxed{-\dfrac{\mathbf{23}}{\mathbf{5}}}$

Implicitly differentiating the equation, we get $3x^2+2xy+x^2y'+y^3+3xy^2y'+4y^3y' = 0$. Putting in $x = -3$ and $y = 2$ we get $23 + 5y' = 0$, from which we get $y' = -23/5$.

Q-3) The sides of a triangle ABC change as a differentiable function of time, but the angle at A always remains $\pi/2$. As a notational convenience we let $A(t)$ =area at time t, and $P(t)$ =perimeter at time t for this triangle. At a particular time t_0 we make the following measurements: $A(t_0) = 30\ cm^2$, $A'(t_0) = 40\ cm^2/sec$, $AB(t_0) = 12\ cm$, $AB'(t_0) = 4\ cm/sec$. Find $P'(t_0)$.

$P(t) = AB + AC + BC, \quad P'(t) = AB' + AC' + BC'$. Since $AB' = 4$ is already given, we need to find AC' and BC' in order to calculate P'.

$A(t) = (1/2)AB \cdot AC, \quad A = 30, \quad AB = 12 \quad \Rightarrow \quad AC = 5.$

$A'(t) = (1/2)(AB' \cdot AC + AB \cdot AC'), \quad A' = 40, \quad AB' = 4 \quad \Rightarrow \quad AC' = 5.$

$AB^2 + AC^2 = BC^2 \quad \Rightarrow \quad BC = 13.$

$2\ AB \cdot AB' + 2\ AC \cdot AC' = 2\ BC \cdot BC' \quad \Rightarrow \quad BC' = 73/13.$

Finally putting these together we get $P'(t_0) = AB' + AC' + BC' = 4 + 5 + 73/13 = 190/13$.

Q-4) Find the volume of the right circular cone of maximal volume that can be inscribed into a sphere of radius R.

Let x be the distance of the base of the cone to the center of the sphere such that when $x > 0$, the base of the cone is below the center. It then follows that the radius r of the base of the cone satisfies $r^2 = R^2 - x^2$. The volume of the cone is $V(x) = \dfrac{\pi}{3}r^2(R + x) = \dfrac{\pi}{3}(R^2 - x^2)(R + x) = \dfrac{\pi}{3}(R^3 - Rx^2 + R^2x - x^3)$, where $-R \leq x \leq R$.

$V'(x) = -\dfrac{\pi}{3}(3x^2 + 2Rx - R^2) = 0$ when $x = -R$ or $x = R/3$.

Since $V = 0$ at the end points, $x = \pm R$, and since $V(x) \geq 0$, the value at $R/3$ must give the global maximum.

$V(\dfrac{R}{3}) = \dfrac{32\pi R^3}{81}.$

Q-5) Consider the function $f(x) = 3x^4 - 16x^3 + 18x^2 - 1$ for $x \in [-1, 4]$.
Find the local min/max points. Find the global min/max points.
Find the concavity and where concavity changes. (you may take $\sqrt{7}$ as 2.6.)
Sketch the curve. You may use the following table.

x	-1		0		0.5		1		2.1		3		4
$f(x)$	36												31
$f'(x)$		$-$		$+$		$+$		$-$		$-$		$+$	
$f''(x)$		$+$		$+$		$-$		$-$		$+$		$+$	
		↘ ⌣		↗ ⌣		↗ ⌢		↘ ⌢		↘ ⌣		↗ ⌣	

$f'(x) = 12x(x-1)(x-3)$, so $f'(x) = 0$ when $x = 0, 1, 3$.

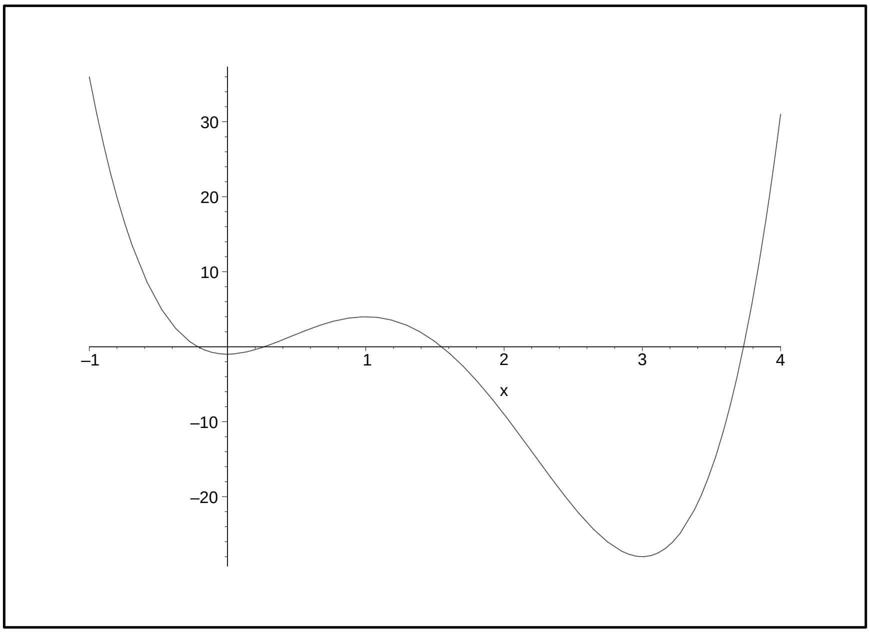
$f''(x) = 12(3x^2 - 8x + 3)$, so $f''(x) = 0$ when $x = \dfrac{4}{3} \pm \dfrac{\sqrt{7}}{3} = 0.5,\ 2.1$ approximately.

By checking the signs of f' and f'' we find that:
$f(0) = -1$ is a local minimum.
$f(1) = 4$ is a local maximum.
$f(3) = -28$ is a local minimum.

We also check the end points: $f(-1) = 36$ and $f(4) = 31$.

Now comparing the above five values we conclude that $f(3) = -28$ is the global minimum and $f(-1) = 36$ is the global maximum.

Here is how the graph looks like (unscaled):



Note that $f(x) = 3x^4 - 16x^3 + 18x^2 - 1 = \left(3\,x^2 - 4\,x - 1\right)\left(x^2 - 4\,x + 1\right)$ from which you can easily find the zeros of f as $2+\sqrt{3} \approx 3.7$, $2-\sqrt{3} \approx 0.2$, $(2+\sqrt{7})/2 \approx 1.5$ and $(2-\sqrt{7})/2 \approx -0.2$. However approximate indications of the roots on the graph is acceptable for the exam purposes.