Date: 18 December 2004, Saturday Instructor: Ali Sinan Sertöz

Math 113 Calculus – Midterm Exam III – Solutions

Q-1) Write the derivatives of the given functions with respect to x. Do not simplify your answer. No partial credits!

a) $f(x) = \ln \sin x + x^{\cos x} + \pi^x + 2^\pi$,

$$f'(x) = \frac{\cos(x)}{\sin(x)} + x^{\cos(x)}\left(-\sin(x)\ln(x) + \frac{\cos(x)}{x}\right) + \pi^x \ln(\pi)$$

b) $f(x) = \dfrac{\cosh x + \cos x}{\ln x}$,

$$f'(x) = \frac{\sinh(x) - \sin(x)}{\ln(x)} - \frac{\cosh(x) + \cos(x)}{(\ln(x))^2\, x}$$

c) $f(x) = (\ln x)^x$,

$$f'(x) = (\ln(x))^x \left(\ln(\ln(x)) + (\ln(x))^{-1}\right)$$

d) $f(x) = e^{\tan x} + (\tan x)^e$,

$$f'(x) = (\sec(x))^2 e^{\tan(x)} + e(\tan(x))^{e-1}(\sec(x))^2$$

Q-2) Evaluate the integral $\displaystyle\int \frac{dx}{\sin x + \cos x + 1}$.

Hint: You may use $u = \tan \frac{1}{2}x$ substitution.

Starting with $u = \tan \frac{1}{2}x$ we get $dx = \dfrac{2}{1+u^2}du$, $\sin x = \dfrac{2u}{1+u^2}$ and $\cos x = \dfrac{1-u^2}{1+u^2}$. Putting these in we get

$$\int \frac{dx}{\sin x + \cos x + 1} = \int \frac{du}{u+1} = \ln|u+1| + C = \ln|1 + \tan\frac{1}{2}x| + C.$$

Q-3) Evaluate the integral $\displaystyle\int \frac{2}{x(x^2+2x+2)}\,dx.$

$$\begin{aligned}
\frac{2}{x(x^2+2x+2)} &= \frac{A}{x} + \frac{Bx+C}{x^2+2x+2} \\
&= \frac{1}{x} - \frac{x+2}{x^2+2x+2} \\
&= \frac{1}{x} - \frac{1}{2}\frac{2x+2}{x^2+2x+2} - \frac{1}{x^2+2x+2} \\
&= \frac{1}{x} - \frac{1}{2}\frac{2x+2}{x^2+2x+2} - \frac{1}{(x+1)^2+1} \\
\int \frac{2}{x(x^2+2x+2)}\,dx &= \ln x - \frac{1}{2}\ln(x^2+2x+2) - \arctan(x+1) + C.
\end{aligned}$$

Q-4) Evaluate the integral $\displaystyle\int \sqrt{1-x^2}\,dx.$

Putting in $x = \sin\theta$ we get $dx = \cos\theta d\theta$, and $\sqrt{1-x^2} = \cos\theta$. Then

$$\begin{aligned}
\int \sqrt{1-x^2}\,dx &= \int \cos^2\theta\,d\theta \\
&= \int \frac{1+\cos 2\theta}{2}\,d\theta \\
&= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \\
&= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C \\
&= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C.
\end{aligned}$$

Q-5) Evaluate the integral $\displaystyle\int x \arcsin x\,dx.$

First we integrate $\arcsin x$ using by-parts with $u = \arcsin x$ to get

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x\arcsin x + \sqrt{1-x^2} + C,$$

where in the last integral we used the substitution $u = 1 - x^2$.

Now we integrate $x\arcsin x$ using by-parts with $u = x$. Then $dv = \arcsin x$ and we know from the above calculation that $v = x\arcsin x + \sqrt{1-x^2}$. This gives

$$\int x\arcsin x\,dx = x^2\arcsin x + x\sqrt{1-x^2} - \int x\arcsin x\,dx - \int \sqrt{1-x^2}\,dx.$$

Solving for $\int x\arcsin x\,dx$ and substituting the value of $\int \sqrt{1-x^2}\,dx$ from the previous question, we find

$$\int x \arcsin x \, dx = \left(\frac{x^2}{2} - \frac{1}{4} \right) \arcsin x + \frac{1}{4} x \sqrt{1 - x^2} + C.$$
