Math 113 Homework 1 - Solutions

Due: 13 October 2005 Thursday class hour for section-2
Due: 14 October 2005 Friday class hour for section-1

Q-1) Find a formula for the sum

$$S(n) = 1 \cdot 2 + 3 \cdot 4 + \cdots + (2n-1)(2n),$$

where $n \in \mathbb{N}^+$. Prove your formula by induction.

Solution:

$$\begin{aligned} S(n) &= 1 \cdot 2 + 3 \cdot 4 + \cdots + (2n-1)(2n) \\ &= \sum_{k=1}^{n} (2k-1)(2k) \\ &= 4\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k \\ &= \frac{4}{3}n^3 + n^2 - \frac{1}{3}n. \end{aligned}$$

Q-2) Find all $x \in \mathbb{R}$ for which we have $|x^2 - 7x + 11| < 1$.

Solution: $|x^2 - 7x + 11| < 1$ means $-1 < x^2 - 7x + 11 < 1$. We then have to solve simultaneously for $0 < x^2 - 7x + 12$ and $x^2 - 7x - 10 < 0$. The common solution set is then $(2,3) \cup (4,5)$.

Q-3) Find the area bounded by $y = |x|$ and $y = 1 - 2x - x^2$.

Solution:

$$\int_{-1/2\,(\sqrt{5})-1/2}^{0} (1 - x - x^2)dx + \int_{0}^{-3/2+1/2\,(\sqrt{13})} (1 - 3\,x - x^2)dx = \frac{5}{12}\sqrt{5} - \frac{19}{6} + \frac{13}{12}\sqrt{13} \approx 1.67.$$

Q-4) Sketch and find the area bounded by the cardioid $f(\theta) = 1 + \sin\theta$ where $0 \le \theta \le 2\pi$.

Solution:

$$\frac{1}{2}\int_{0}^{2\,\pi} (1 + \sin(\theta))^2\, d\theta = \frac{3\pi}{2}.$$

Q-5) Sketch the region bounded by the line $y = 10 - x$ and the curve $y = 9/x$.

i) Find the area of this region. Here you may take $\int_1^9 (1/x)dx \approx 2.2$.

ii) Find the volume obtained by revolving this region around the x-axis.

iii) Find the volume obtained by revolving this region around the y-axis.

Solution:

i)

$$\int_1^9 (10 - x - 9/x)\, dx = 40 - 9(2.2) \approx 20.22.$$

ii)

$$\pi \int_1^9 \left((10 - x)^2 - (9/x)^2\right) dx = \frac{512\pi}{3}.$$

iii) $\dfrac{512\pi}{3}$ due to symmetry!

`comments and questions to sertoz@bilkent.edu.tr`