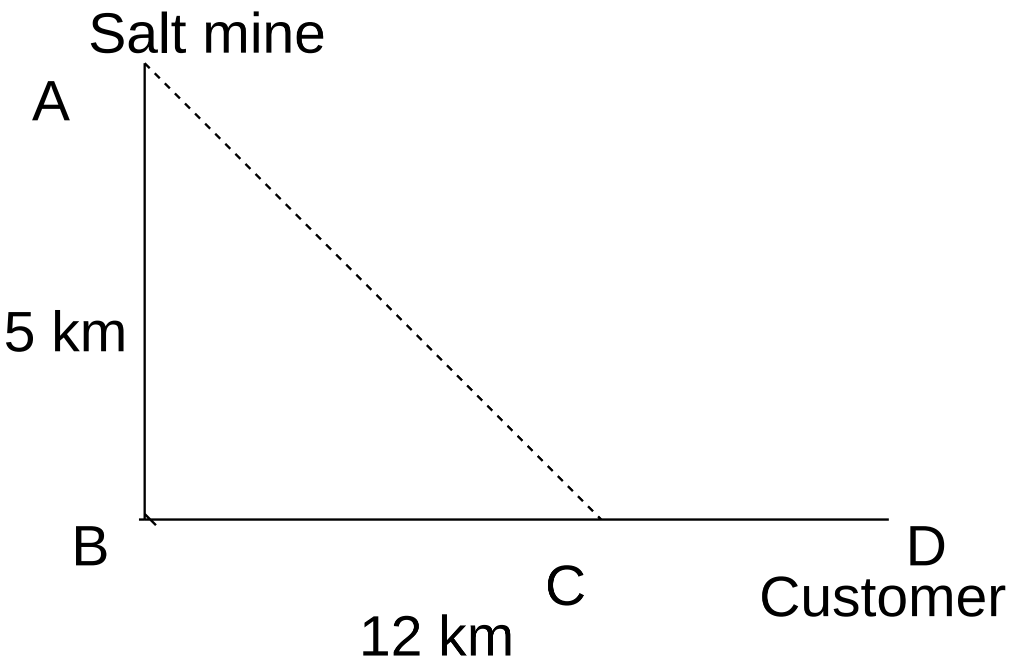
Math 113 Homework 3 – Solutions

Due: 15 November 2005 Tuesday.

Q-1) We have a salt mine 5km inland from a straight coast line, (see figure.) Our customer is located 12km away along the coast. The cost of transportation along the coast is α times more expensive than that on land. Find the optimal path of transportation which minimizes our cost. Note that $\alpha \geq 0$ and the answer depends on α.



Solution:

Denote the length $|BC|$ by x. Then $|CD| = 12 - x$, and the cost function to minimize is

$$f(x) = \sqrt{25 + x^2} + \alpha(12 - x), \quad 0 \leq x \leq 12.$$

We find that

$$f'(x) = \frac{x}{\sqrt{25 + x^2}} - \alpha$$

and that

$$f'(x) = 0 \quad \text{if and only if} \quad (1 - \alpha^2)x = 25\alpha^2.$$

Now we have several cases depending on the value of $\alpha \geq 0$.

Case 1: $\alpha \geq 1$.
Then there are no critical points so we check only the end points of the domain of f.
$f(0) = 5 + 12\alpha \geq 5 + 12 = 17$ since $\alpha \geq 1$.
$f(12) = 13$.
So $f(12)$ is the minimum value.

Case 2: $0 \le \alpha < 1$.
Then $x_0 = \dfrac{5\alpha}{\sqrt{1-\alpha^2}}$ is the only nonnegative critical point. This critical point will be useful for us if it is in the domain of f, i.e. we want $x_0 \le 12$. This forces $\alpha \le \dfrac{12}{13}$.

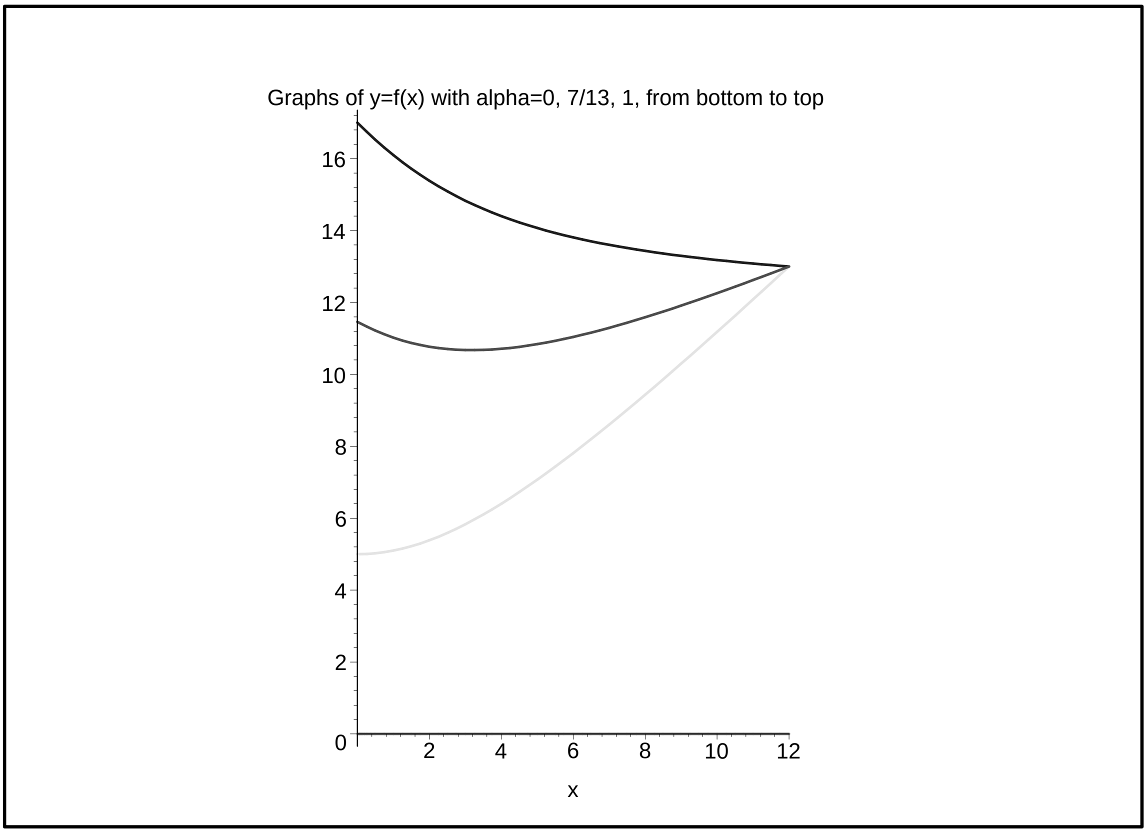
Case 2.1: $\dfrac{12}{13} \le \alpha \le 1$.
In this case $x_0 \ge 12$ so we again check only the end points.
$f(0) = 5 + 12\alpha \ge 5 + \dfrac{144}{13} \ge 16$.
$f(12) = 13$.
So again the minimum value is $f(12)$.

Case 2.2: $0 < \alpha < \dfrac{12}{13}$.
$f(0) = 5 + 12\alpha$,
$f(12) = 13$,
$f(x_0) = 5\sqrt{1-\alpha^2} + 12\alpha$.
By direct computation we check that $f(x_0)$ is the minimum value.

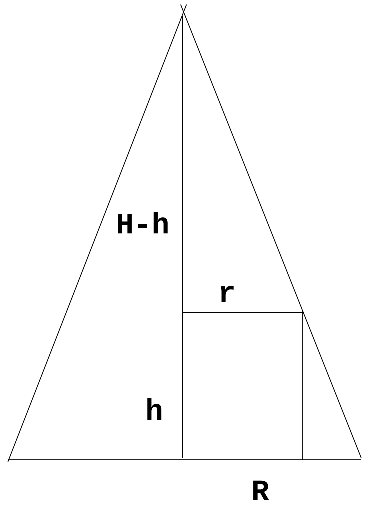
Case 2.3: $\alpha = 0$.
In this case $x_0 = 0$, $f(0) = 5$ and $f(12) = 13$. So the minimum value is $f(0)$.

Summary of cases:
If $\alpha = 0$, then the minimum value is $f(0) = 5$.
If $0 < \alpha < \dfrac{12}{13}$, then the minimum occurs at the point $x_0 = \dfrac{5\alpha}{\sqrt{1-\alpha^2}}$. Check that $0 < x_0 < 12$ and $f(x_0) = 5\sqrt{1-\alpha^2} + 12\alpha$.
If $\alpha \ge \dfrac{12}{13}$, then the minimum value is $f(12) = 13$.

Here are a few sample graphs with different α's:



Q-2) (Page 195, Exercise 12) Given a right circular cone with radius R and altitude H. Find the radius and altitude of the right circular cylinder of largest lateral surface area that can be inscribed in the cone.

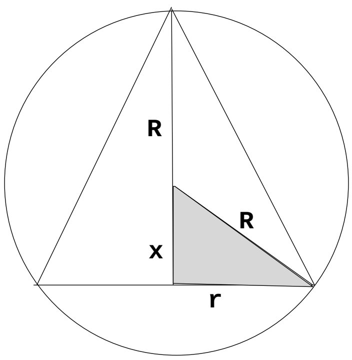


Solution: From similar triangles we find $h = H - \frac{H}{R}r$. The function to maximize is

$$f(r) = 2\pi H \left(r - \frac{1}{R}r^2 \right), \quad 0 \le r \le R.$$

$f'(r) = 0$ when $r = R/2$. Then $h = H/2$. Since $f(0) = f(R) = 0$, the critical point gives the maximum value $f(R/2) = HR\pi/4$.

Q-3) (Page 195, Exercise 14) Given a sphere of radius R. Compute, in terms of R, the radius r and the altitude h of the right circular cone of maximum volume that can be inscribed in this sphere.

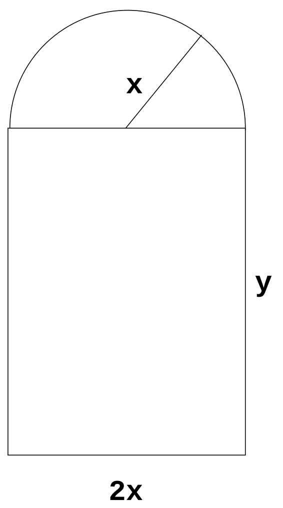


Solution: From the above right triangle we find $r^2 = R^2 - x^2$. Let the height of the cone be h. Then $h = R + x$. The volume function to maximize is

$$V(x) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(R^3 + R^2x - Rx^2 - x^3), \quad -R \le x \le R.$$

We find that $V'(x) = 0$ when $x = R/3$ interior the domain. Since $V(-R) = V(R) = 0$ and $V(R/3) = 8\pi R^3/81$, this critical point gives the maximum volume. In that case $r = 2\sqrt{2}R/3$ and $h = 4R/3$.

Q-4) (Page 196, Exercise 23) A window is to be made in the form of a rectangle surmounted by a semicircle with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass, and the semicircular portion is to be of a colored glass admitting only half as much light per square foot as the clear glass. The total perimeter of the window frame is to be a fixed length P. Find, in terms of P, the dimensions of the window which will admit the most light.



Solution:

Since $P = 2x + 2y + \pi x$, we must have $y = \dfrac{1}{2}(P - (2+\pi)x)$ and $0 \leq x \leq \dfrac{P}{2+\pi}$.

The function to maximize is

$$f(x) = (2xy) + \frac{1}{2}(\frac{\pi x^2}{2}) = Px - \frac{1}{4}(3\pi + 8)x^2, \quad 0 \leq x \leq \frac{P}{2+\pi}.$$

We see that $f'(x) = 0$ when $x = 2P/(3\pi+8)$, and since $f(0) = f(\frac{P}{2+\pi}) = 0$ and $f(2P/(3\pi+8)) = P^2/(3\pi+8)$, this critical point gives the maximum value. In that case the base of the rectangle is $2x = 4P/(3\pi + 8)$ and the height is $y = P(\pi+4)/(6\pi+16)$.

Q-5) (Page 196, Exercise 25) Given n real numbers $a_1, \ldots, a_n$. Prove that the sum $\sum_{k=1}^{n}(x - a_k)^2$ is smallest when x is the arithmetic mean of $a_1, \ldots, a_n$.

Solution: Let $f(x) = \sum_{k=1}^{n}(x - a_k)^2$ for all real x. f is continuous, is always nonnegative and becomes unbounded as $|x|$ increases. So it must have a global minimum.

$f'(x) = 2(nx - (a_1 + \cdots + a_n))$ and $f'(x) = 0$ when $x = (a_1 + \cdots + a_n)/n$. Since this is the only critical point, it must give the global minimum.

`Please send comments and questions to sertoz@bilkent.edu.tr`