# Math 113 – Homework 4 – Solutions

Due: 22 November 2005 Tuesday.

**Q-1)** Let $f(x) = x^x$ and $g(x) = x^{x^x}$. Find $f'(x),\ f''(x),\ g'(x),\ g''(x)$. Simplify your answers for ease of reading.

$f(x) = \exp(\ln f(x)) = \exp(\ln x^x) = \exp(x \ln x)$.

$f'(x) = (\exp(x \ln x))' = \exp(x \ln x) \cdot (\ln x + 1)$. So $f'(x) = x^x(\ln x + 1)$.

$f''(x) = (x^x(\ln x + 1))' = x^x(\ln x + 1)^2 + x^{x-1}$.

Similarly $g(x) = \exp(\ln x^{x^x}) = \exp(f(x) \ln x)$.

$g'(x) = x^{x^x}(f'(x) \ln x + f(x)/x) = x^{x^x}\left(x^x(\ln x + 1)\ln x + x^{x-1}\right)$

$g''(x) = x^{x^x}\left(x^x(\ln x + 1)\ln x + x^{x-1}\right)^2 + x^{x^x}\left(x^x(\ln x + 1)^2 \ln x + x^{x-1}(3\ln x + 2) - x^{x-2}\right)$.

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**Q-2)** Let $I_{n,m} = \displaystyle\int x^n (\ln x)^m \, dx$, where $n, m$ are nonnegative integers. Find a formula for $I_{n,m}$ and prove your claim.

Integrating $I_{n,m}$ by parts we immediately find the recursive formula

$$I_{n,m} = \frac{1}{n+1} x^{n+1} (\ln x)^m - \frac{m}{n+1} I_{n,m-1}.$$

We then guess the formula

$$I_{n,m} = \frac{m!\, x^{n+1}}{n+1} \sum_{k=0}^{m} \frac{(-1)^{m-k} (\ln x)^k}{(n+1)^{m-k} k!} \ + \ C.$$

To prove the formula we do induction on $N = m + n$. By direct inspection we see that the formula holds for $N = 0$ i.e. for $n = m = 0$. Now assume the formula for $N - 1$ and let $n$ and $m$ be nonnegative integers with $n + m = N$. We have

$$\begin{aligned}
I_{n,m} &= \frac{1}{n+1} x^{n+1} (\ln x)^m - \frac{m}{n+1} I_{n,m-1} \\
&= \frac{1}{n+1} x^{n+1} (\ln x)^m - \frac{m}{n+1} \frac{(m-1)!\, x^{n+1}}{n+1} \sum_{k=0}^{m-1} \frac{(-1)^{m-1-k} (\ln x)^k}{(n+1)^{m-1-k} k!} \ + \ C \\
&= \frac{m!}{(n+1)m!} x^{n+1} (\ln x)^m + \frac{m!\, x^{n+1}}{n+1} \sum_{k=0}^{m-1} \frac{(-1)^{m-k} (\ln x)^k}{(n+1)^{m-k} k!} \ + \ C \\
&= \frac{m!\, x^{n+1}}{n+1} \sum_{k=0}^{m} \frac{(-1)^{m-k} (\ln x)^k}{(n+1)^{m-k} k!} \ + \ C,
\end{aligned}$$

and this proves the formula.

**Q-3)** Let $I_{a,b,c} = \int \frac{dx}{ax^2+bx+c}$, where $a, b, c$ are real numbers with $\Delta = b^2 - 4ac < 0$. Find $I_{a,b,c}$.

First let $\alpha = \frac{\sqrt{4ac-b^2}}{2a}$. Then

$$\begin{aligned} I_{a,b,c} &= \frac{1}{a}\int \frac{dx}{(x+\frac{b}{2a})^2+\alpha^2} \\ &= \frac{1}{a\alpha}\arctan(\frac{x}{\alpha}+\frac{b}{2a\alpha})+C \\ &= \frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)+C. \end{aligned}$$

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**Q-4)** Find $\int \frac{x\ dx}{x^2+x+1}$.

$$\begin{aligned} \int \frac{x\ dx}{x^2+x+1} &= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\ dx - \frac{1}{2}\int \frac{1}{x^2+x+1}\ dx \\ &= \frac{1}{2}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+C. \end{aligned}$$

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**Q-5)** Find $\int \frac{x^2\ dx}{\sqrt{1-x^2}}$.

Putting $x = \sin\theta$ we get

$$\begin{aligned} \int \frac{x^2\ dx}{\sqrt{1-x^2}} &= \int \sin^2\theta\ d\theta = \frac{1}{2}\ \theta - \frac{1}{4}\sin 2\theta + C \\ &= \frac{1}{2}\ \theta - \frac{1}{2}\sin\theta\cos\theta = \frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2}+C. \end{aligned}$$

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`Comments and questions to sertoz@bilkent.edu.tr`