Date: 2 December 2005, Friday

Math 113 Calculus – Second Midterm Exam – Solutions

Q-1) Do not simplify your answers in (a), (b), (c). No partial credits!

a) $f(x) = x^x + (\ln x)^{\cosh x} = e^{x \ln x} + e^{\cosh x \ln \ln x}$,

$f'(x) = x^x(\ln x + 1) + (\ln x)^{\cosh x}(\sinh x \ln \ln x + \cosh x \frac{1}{\ln x}\frac{1}{x})$.

b) $f(x) = \arctan\left(\sqrt{1-x^2}\right) \cdot \sec\sqrt{1+x^2}$,

$f'(x) = \frac{1}{1+(1-x^2)}\frac{-2x}{2\sqrt{1-x^2}} \sec\sqrt{1-x^2} + \arctan\sqrt{1-x^2}\ \sec\sqrt{1-x^2}\ \tan\sqrt{1-x^2}\ \frac{x}{\sqrt{1-x^2}}$.

c) $f(x) = \cos x \int_{\sec x}^{\tan x} \sin t\, dt$,

$f'(x) = -\sin x \int_{\sec x}^{\tan x} \sin t\, dt + \cos x \left(\sin(\tan x)\sec^2 x - \sin(\sec x)\sec x \tan x\right)$.

d) Let $f(x) = g^{-1}(x)$ for $1 \le x \le 5$, where $g(x) = x^3 - 3x^2 + 5$. Find the slope of the tangent line to the curve $y = f(x)$ at the point $(3, 1)$.

Solution:

$f(3) = 1 \Leftrightarrow g(1) = 3$. The required slope is $f'(3)$.

$f'(3) = f'(g(1)) = 1/g'(1)$.

$g'(x) = 3x^2 - 6x$, $g'(1) = -3$, $f'(3) = -1/3$.

Q-2) Find all values of α, $\beta \in \mathbb{R}$ so that if f is defined as

$$f(x) = \begin{cases} x^\alpha \sin\frac{1}{x} & \text{if } x > 0, \\ \beta & \text{if } x \le 0. \end{cases}$$

then
(i) f is continuous at $x = 0$.
(ii) f is differentiable at $x = 0$.

Solution:

(i) Since $-x^\alpha \le x^\alpha \sin\frac{1}{x} \le x^\alpha$, $\lim_{x\to 0^+} f(x)$ exists if and only if $\alpha > 0$. In that case the limit is zero.

Hence f is continuous at $x = 0$ when $\alpha > 0$ and $\beta = 0$.

(ii) For f to be differentiable at $x = 0$, it must first be continuous there. So in particular $\beta = 0$.

$\lim_{x\to 0^+} \dfrac{f(x) - f(0)}{x} = \lim_{x\to 0^+} x^{\alpha-1} \sin \dfrac{1}{x}$ exists if and only if $\alpha > 1$. In that case the limit is zero.

On the other hand with $\beta = 0$ we have $\lim_{x\to 0^-} \dfrac{f(x) - f(0)}{x} = 0$.

Hence f is differentiable at $x = 0$ when $\alpha > 1$ and $\beta = 0$.

Q-3) Find the minimum and the maximum values of the function $f(x)$ on $[0, 8]$, where

$$f(x) = \begin{cases} x^3 - 4x^2 - 3x + 8 & \text{if } \ 0 \leq x \leq 5, \\ x^2 - 14x + 63 & \text{if } \ 5 \leq x \leq 8. \end{cases}$$

Solution:

For $0 \leq x \leq 5$, $f'(x) = 3x^2 - 8x - 3 = 0$, $x = 3$ is the solution in the domain.

For $5 \leq x \leq 8$, $f'(x) = 2x - 14 = 0$, $x = 7$

We calculate $f(0) = 8$, $f(3) = -10$, $f(5) = 18$, $f(7) = 14$, $f(8) = 15$.

Therefore minimum of f is -10, and the maximum is 18.

Q-4) Assume that $f'(x) = \dfrac{x^2 + 2}{((x-1)(x-2))^2}$, and $f(0) = 0$. Find $f(3)$.

By partial fractions we find

$$\frac{x^2 + 2}{((x-1)(x-2))^2} = \frac{8}{x-1} + \frac{3}{(x-1)^2} - \frac{8}{x-2} + \frac{6}{(x-2)^2}.$$

Solution:

Integrating this we find $f(x) = 8 \ln |x - 1| - \dfrac{3}{x-1} - 8 \ln |x - 2| - \dfrac{6}{x-2} + C$.

$f(0) = 0$ gives $C = 8 \ln 2 - 6$, and we find $f(3) = 16 \ln 2 - \dfrac{27}{2}$.

Q-5) Find $\displaystyle\int x^2 \ \arctan x \, dx$.

First by using by parts with $u = \arctan x$ and $dv = x^2 dx$ we obtain

$$\int x^2 \ \arctan x \, dx = \frac{1}{3} x^3 \arctan x - \frac{1}{3} \int \frac{x^3}{1 + x^2} \, dx.$$

Solution:

We have $\dfrac{x^3}{1+x^2} = x - \dfrac{1}{2}\,\dfrac{2x}{1+x^2}$. Integrating this we find $\displaystyle\int \frac{x^3}{1+x^2}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2) + C$. Putting this back in the above integral we finally get

$$\int x^2\ \arctan x\,dx = \frac{1}{3}x^3\arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(1+x^2) + C.$$
