Date: January 11, 2008 Friday
Time: 12:15-14:15
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NAME:..

STUDENT NO:..

Math 113 Calculus – Final Exam – Solutions

Q-1) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two uniformly continuous functions.

Prove or disprove: The function $f \circ g$ is uniformly continuous on $\mathbb{R}$.

Solution: The function $f \circ g$ is uniformly continuous on $\mathbb{R}$.

Let $\epsilon > 0$ be chosen at random. Since f is uniformly continuous on $\mathbb{R}$, there exists a $\delta_0 > 0$ such that for all $y_1, y_2 \in \mathbb{R}$ with $|y_1 - y_2| < \delta_0$ we must have $|f(y_1) - f(y_2)| < \epsilon$. Now using the uniform continuity of g on $\mathbb{R}$, we can find a $\delta > 0$ such that for all $x_1, x_2 \in \mathbb{R}$ with $|x_1 - x_2| < \delta$, we must have $|g(x_1) - g(x_2)| < \delta_0$.

It is now clear that for all $x_1, x_2 \in \mathbb{R}$ with $|x_1 - x_2| < \delta$, we have $|(f \circ g)(x_1) - (f \circ g)(x_2)| < \epsilon$.

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Q-2) Find the limit

$$\lim_{x\to 0}\frac{\sin x\sinh x - x^2\cos x}{\cos x\cosh x - 1 + x^4}$$

Solution: Let $N(x) = \sin x\sinh x - x^2\cos x$ and $D(x) = \cos x\cosh x - 1 + x^4$. Using Taylor's theorem we have

$$\begin{aligned}
N(x) &= \left(x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right)\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right) - x^2\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) \\
&= \frac{1}{2}x^4 - \frac{19}{360}x^6 + \cdots \\
D(x) &= \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots\right) - 1 + x^4 \\
&= \frac{5}{6}x^4 + \frac{1}{2520}x^8 + \cdots
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0}\frac{N(x)}{D(x)} &= \lim_{x\to 0}\frac{\frac{1}{2}x^4 - \frac{19}{360}x^6 + \cdots}{\frac{5}{6}x^4 + \frac{1}{2520}x^8 + \cdots} \\
&= \lim_{x\to 0}\frac{\frac{1}{2} - \frac{19}{360}x^2 + \cdots}{\frac{5}{6} + \frac{1}{2520}x^4 + \cdots} \\
&= \frac{3}{5}\,.
\end{aligned}$$

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Q-3) Find the constants A and B such that

$$\int_2^8 \frac{\ln x}{(x+1)^2}\,dx = A\ln 2 + B\ln 3.$$

Solution: Start with integration by parts letting $u = \ln x$ and $dv = dx/(x+1)^2$. Then $du = dx/x$ and $v = -1/(x+1)$. We get

$$\int_2^8 \frac{\ln x}{(x+1)^2}\,dx = \left(-\frac{\ln x}{x+1}\bigg|_2^8\right) + \int_2^8 \frac{dx}{x(x+1)} = \int_2^8 \frac{dx}{x(x+1)}\,.$$

(Check it!) Next using partial fractions technique we find

$$\int_2^8 \frac{dx}{x(x+1)} = \int_2^8 \frac{dx}{x} - \int_2^8 \frac{dx}{1+x} = 2\ln 2 - \ln 3.$$

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Q-4) For $x > 0$, define a function $f(x) = 3x^2 + \dfrac{2A}{x^3}$, where A is a positive constant.

Find the smallest value of A such that $f(x) \geq 45$ for all $x > 0$.

Solution: We must arrange A such that the minimum value of f is 45. For this we find

$$f'(x) = \frac{6}{x^4}(x^5 - A) = 0.$$

Let B be the positive number with $B^5 = A$. Then $x = B$ is the only critical point for f. Since f approaches to infinity as x approaches to the boundary points, i.e. as $x \to 0+$ and as $x \to \infty$), $x = B$ must give the global minimum point. We set this global minimum value to 45 to find B and hence A.

$$f(B) = 5B^2 = 45, \quad B = 3, \quad A = 243.$$

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Q-5) $f : (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ is a differentiable function which is always positive, and it satisfies the identity

$$f^2(x) - 1 = 2\int_0^x f^2(t)\sec^2 t\,dt, \text{ for all } x \in (-\pi/2, \pi/2).$$

Find explicitly what $f(x)$ is.

Solution: First we observe that $f(0) = 1$. Then differentiating both sides of the identity with respect to x and using the fundamental theorem of calculus we find

$$\begin{aligned}
2f(x)f'(x) &= 2f^2(x)\sec^2 x \\
\frac{f'(x)}{f(x)} &= \sec^2 x \\
(\ln f(x))' &= (\tan x)' \\
\ln f(x) - \ln f(0) &= \tan x - \tan 0 \\
f(x) &= e^{\tan x}.
\end{aligned}$$

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Bonus:) Evaluate the integral

$$\int_0^1 (\arcsin x)^2 \, dx.$$

Recall that $\dfrac{d \arcsin x}{dx} = \dfrac{1}{\sqrt{1-x^2}}$ for $|x| < 1$.

Solution: First we attack the indefinite integral with by-parts letting $u = (\arcsin x)^2$, getting

$$\int (\arcsin x)^2 \, dx = x(\arcsin x)^2 - 2 \int \arcsin x \, \frac{x}{\sqrt{1-x^2}} \, dx.$$

For the second integral we again use by-parts with $u = \arcsin x$ to get

$$\int \arcsin x \, \frac{x}{\sqrt{1-x^2}} \, dx = -\sqrt{1-x^2} \arcsin x + x + C.$$

Putting these together we find

$$\int (\arcsin x)^2 \, dx = x \left(\arcsin x\right)^2 + 2\sqrt{1-x^2} \arcsin x - 2x + C$$

and finally

$$\int_0^1 (\arcsin x)^2 \, dx = \frac{\pi^2}{4} - 2 \approx 0.467401101.$$
