Calculus 113 Homework 4

Due date: 2 November 2007 Friday

Please take your homework solutions to room SA144, Ali Adalı's office.

Q-1) Find the largest $\delta > 0$ satisfying the property that for all $x, y \in [0, 100]$ with $|x - y| < \delta$, we have $|x^2 - y^2| < 1$. Show that no such $\delta > 0$ exists if we choose $x, y \in [0, \infty)$.

Solution: When $0 \le y < x < 1$, we always have $|x^2 - y^2| < 1$. So let $x \ge 1$ and $y = x - h$ where $h \ge 0$. Then $|x^2 - y^2| < 1$ means $h^2 - 2xh + 1 > 0$. We find that this inequality holds for all h with $0 \le h < x - \sqrt{x^2 - 1} = 1/(x + \sqrt{x^2 - 1})$. This is a decreasing function of x and will take its minimum at $x = 100$. Let $\delta = 100 - \sqrt{9999}$. Then for any $x, y \in [0, 100]$, we just showed that $|x - y| < \delta$ implies $|x^2 - y^2| < 1$. If we consider the same problem on the interval $[0, \infty)$, then there is no positive minimum value for $1/(x + \sqrt{x^2 - 1})$, so the function cannot be uniformly continuous there.

Q-2-a) Find a function f which is continuous and bounded but not uniformly continuous on $(0, 1]$.

Solution: $f(x) = \sin(1/x)$ for $x \in (0, 1]$.

Q-2-b) Find a function f which is continuous and bounded but not uniformly continuous on $[0, \infty)$.

Solution: $f(x) = \sin x^2$ for $x \in [0, \infty)$.

Q-3-a) For any non-negative integer $n \in \mathbb{N}$, find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{(n)}(x)$ exists and is continuous for all $x \in \mathbb{R}$, but $f^{(n+1)}(0)$ does not exist.

Solution:

$$f_n(x) = \begin{cases} \dfrac{x^{n+1}}{(n+1)!} & \text{if } x \ge 0, \\[2ex] -\dfrac{x^{n+1}}{(n+1)!} & \text{if } x < 0. \end{cases}$$

Where $n = 0, 1, 2, \ldots$. Observe that $f'_{(n+1)}(x) = f_n(x)$.

Q-3-b) For any non-negative integer $n \in \mathbb{N}$, find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f^{(n)}(x)$ exists for all $x \in \mathbb{R}$, but $f^{(n)}(x)$ is not continuous at $x = 0$.

Solution:

$$f_n(x) = \begin{cases} x^{2n} \sin(1/x) & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Where $n = 0, 1, 2, \ldots$.

Q-4-a) Consider the function

$$f(x) = \begin{cases} x^2 \sin^2(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that $x = 0$ is a local minimum for f but f is neither decreasing to the left of 0 nor increasing to the right of it.

Solution: In any right or left neighborhood of 0, we can find points $x_1 < x_2 < x_3$ such that $f(x_1) = f(x_3) = 0$ and $f(x_2) = 1$.

Q-4-b) Consider the function

$$f(x) = \begin{cases} \alpha x + x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

where $0 < \alpha < 1$. Show that $f'(0) = \alpha > 0$ but f is not increasing on any open interval containing 0.

Solution: In every neighborhood of 0 there are infinitely many points where the graph of the function lies on the line $y = \alpha x$ and in between each such pair of points there are points where the graph is above that line.

Please forward any comments or questions to `sertoz@bilkent.edu.tr`