Calculus 113 Homework 6

Solutions

Q-1) Find a recursive formula for $I_{m,n} = \displaystyle\int x^m \ln^n x \; dx$, and use it to evaluate $\displaystyle\int_1^2 x^3 \ln^2 x \, dx$.

Solution: By applying by-parts with $u = \ln^n x$ we find immediately that

$$I_{m,n} = \frac{x^{m+1}}{m+1} \ln^n x - \frac{n}{m+1} \, I_{m,n-1}.$$

This suggests that we can explicitly evaluate $I_{m,n}$. First check that for any integer $m \geq 0$,

$$\begin{aligned}
I_{m,0} &= \frac{x^{m+1}}{m+1} + C, \\
I_{m,1} &= \frac{x^{m+1}}{m+1} \ln x - \frac{x^{m+1}}{(m+1)^2} + C, \\
I_{m,2} &= \frac{x^{m+1}}{m+1} \ln^2 x - \frac{2x^{m+1}}{(m+1)^2} \ln x + \frac{2x^{m+1}}{(m+1)^3} + C, \\
I_{m,3} &= \frac{x^{m+1}}{m+1} \ln^3 x - \frac{3x^{m+1}}{(m+1)^2} \ln^2 x + \frac{3 \cdot 2x^{m+1}}{(m+1)^3} \ln x - \frac{3 \cdot 2x^{m+1}}{(m+1)^4} + C.
\end{aligned}$$

From these we guess that for any integers $m, n \geq 0$, we should have:

$$I_{m,n} = x^{m+1} \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \frac{k!}{(m+1)^{k+1}} \ln^{n-k} x + (-1)^n \frac{n! x^{m+1}}{(m+1)^{n+1}} + C.$$

We can check the validity of this expression either by induction or by simply taking the derivative of both sides.

Using this we see that

$$\int x^3 \ln^2 x \, dx = \frac{1}{4} x^4 \ln^2 x - \frac{1}{8} x^4 \ln x + \frac{1}{32} x^4 + C$$

and

$$\int_1^2 x^3 \ln^2 x \, dx = 4 \ln^2 2 - 2 \ln 2 + \frac{15}{32} \approx 1.004267695.$$

Q-6) Let $f : (a, b) \longrightarrow \mathbb{R}$ be a uniformly continuous function. Show that there exists a continuous function $F : [a, b] \longrightarrow \mathbb{R}$ such that $F(x) = f(x)$ for all $x \in (a, b)$.

Solution: Assume that $\lim_{x \to b-} f(x)$ and $\lim_{x \to a+} f(x)$ exist. Then define the function $F(x)$ on $[a, b]$ as follows:

$$F(x) = \begin{cases} f(x), & \text{if } x \in (a, b), \\ \lim_{x \to b-} f(x) & \text{if } x = b, \\ \lim_{x \to a+} f(x) & \text{if } x = a. \end{cases}$$

Clearly $F(x)$ is continuous on $[a, b]$, and being continuous on a closed and bounded interval, it is uniformly continuous.

It remains to show that these limits exist. We will show that the nonexistence of any of these limits contradicts the uniform continuity of f on (a, b).

First recall that in general $\lim_{x \to b-} f(x)$ exists means

$$\exists L \in \mathbb{R}\ \forall \epsilon > 0\ \exists \delta > 0 \text{ such that } \forall x \in B_\delta(b) \cap (a, b) \text{ we have } |f(x) - L| < \epsilon,$$

where $B_\delta(b) = \{t \in \mathbb{R} |\ |t - b| < \delta\}$. In particular, if we choose any two points $x, y \in B_\delta(b) \cap (a, b)$, we should have

$$|f(x) - f(y)| = |(f(x) - L) - (f(y) - L)| \leq |f(x) - L| + |f(y) - L| < 2\epsilon.$$

We thus showed that if $\lim_{x \to b-} f(x)$ exists, then

$$\exists L \in \mathbb{R}\ \forall \epsilon > 0\ \exists \delta > 0 \text{ such that } \forall x, y \in B_\delta(b) \cap (a, b) \text{ we have } |f(x) - f(y)| < 2\epsilon.$$

Negating this we find that if $\lim_{x \to b-} f(x)$ does not exist, then

$$\forall L \in \mathbb{R}\ \exists \epsilon > 0 \text{ such that } \forall \delta > 0\ \exists x, y \in B_\delta(b) \cap (a, b) \text{ such that } |f(x) - f(y)| \geq 2\epsilon > \epsilon.$$

Here note two things. First note that L does not show up in the concluding statement, and next observe that $x, y \in B_\delta(b) \cap (a, b)$ implies $|x - y| < \delta$. So we actually showed that if $\lim_{x \to b-} f(x)$ does not exist, then

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0\ \exists x, y \in (a, b) \text{ with } |x - y| < \delta \text{ such that } |f(x) - f(y)| \geq \epsilon.$$

On the other hand f is uniformly continuous so we do have

$$\forall \epsilon > 0\ \exists \delta > 0 \text{ such that } \forall x, y \in (a, b) \text{ with } |x - y| < \delta \text{ we have } |f(x) - f(y)| < \epsilon.$$

The last two statements obviously claim exactly the opposite things.

This contradiction shows that $\lim_{x \to b-} f(x)$ exists. Similarly $\lim_{x \to a+} f(x)$ exists, and that completes the solution.

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