Date: November 24, 2007, Saturday NAME:..
Time: 10:00-12:00
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Math 113 Calculus – Midterm Exam II – Solutions

Q-1) Prove or disprove: The function $f(x) = \sin x^2$ is uniformly continuous on $\mathbb{R}$.

Solution: This is clearly wrong! Assume that $f(x)$ is uniformly continuous on $\mathbb{R}$. Take $\epsilon = 1/2$. In fact any $0 < \epsilon < 1$ will do.

First observe that $\sqrt{(n+1)\pi} - \sqrt{n\pi} = \dfrac{\pi}{\sqrt{(n+1)\pi} + \sqrt{n\pi}}$, and
$\sqrt{n\pi} < \sqrt{(n+1/2)\pi} < \sqrt{(n+1)\pi}$.

Assume now that, since f is uniformly continuous, there is a $\delta > 0$ such that for every x, y with $|x - y| < \delta$ we will have $|f(x) - f(y)| < \epsilon$.

Take $x = \sqrt{n\pi}$, $y = \sqrt{(n+1/2)\pi}$ where n is such that

$$\sqrt{(n+1)\pi} - \sqrt{n\pi} = \frac{\pi}{\sqrt{(n+1)\pi} + \sqrt{n\pi}} < \delta.$$

Then $|x - y| < \delta$, $f(x) = 0$ and $f(y) = (-1)^n$, so $|f(x) - f(y)| = 1$ which is not smaller than our chosen ϵ. This contradiction shows that f is not uniformly continuous on $\mathbb{R}$.

NAME: STUDENT NO:

Q-2) Let $f(x)$ be differentiable on $[a,b]$, except possibly at $x_0 \in (a,b)$. Assume that $\lim_{x\to x_0} f'(x) = 2007$.

Prove or disprove: $f(x)$ is differentiable also at x_0 and in fact $f'(x_0) = 2007$.

Solution: This is clearly correct!

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}, \text{ (definition of derivative at } x_0\text{)} \\ &= \lim_{x\to x_0} f'(c), \text{ (for some } x_0 \text{ between } x \text{ and } x_0\text{; MVT.)} \\ &= \lim_{c\to x_0} f'(c), \text{ (since } c \text{ is between } x \text{ and } x_0\text{.)} \\ &= 2007, \text{ (as given in the problem.).} \end{aligned}$$

Remark: This problem was intended to be an easy problem which I solved twice in class. In haste I forgot to state here that f is defined and continuous on $[a,b]$. As stated above the statement is false; think about any step function where x_0 is any jump point. But if you solve the problem assuming continuity of f on all of $[a,b]$, I will certainly accept your solution.

NAME: STUDENT NO:

Q-3) Our intelligent dog Elvis can run 5 m/min on the shore and can swim at 3 m/min. He stands at a point A on the shore. We assume that the shore is a straight line. Elvis wants to get to a tennis ball which is floating still at a point B on the sea. The perpendicular line from B to the shore intersects the shore line at C. The distances are as follows: $AC = 12m$ and $BC = 5m$. Obviously Elvis will instinctively choose a point D on the line AC, then run up to D, and swim the distance DB so as to minimize his time to reach the ball from where he is standing. Assuming that Elvis the dog intuitively makes correct choices, tell us precisely where D is.

Solution: Let D be x meters away from C towards A, and let $f(x)$ denote the time in minutes if Elvis chooses this point as D. Then the function to minimize is

$$f(x) = \frac{12-x}{5} + \frac{\sqrt{25+x^2}}{3}, \ 0 \le x \le 12.$$

We find that $f'(x) = -\dfrac{1}{5} + \dfrac{x}{3\sqrt{25+x^2}}$. The only critical point in the given interval of the function is $x = 15/4$.

We check that:

$f(0) = \dfrac{61}{15} > 4,$

$f(12) = \dfrac{13}{3} > 4,$

$f(15/4) = \dfrac{17}{10} + \dfrac{\sqrt{149}}{6} < 1.7 + 13/6 < 1.7 + 2.2 < 4,$

so the minimum does occur at D which is 15/4 meters away from C towards A.

If you do not want to calculate $f(15/4)$, then you can easily calculate

$$f''(x) = \frac{25}{3(25+x^2)^{3/2}} > 0$$

so the given critical point is a local minimum. Since it is the only critical point, it must give the global minimum. This way you do not calculate f at the end points either.

NAME: STUDENT NO:

Q-4) The radius and height of a right circular cylinder of fixed volume of 6860π cubic meters are changing as functions of time. Find how much the radius is changing when the height is $35m$ and is decreasing at a rate of $13m/min$.

Solution: $6860\pi = \pi r^2(t)h(t)$, where $r(t)$ and $h(t)$ are the radius and height at time t, respectively. Take the derivative of this equation with respect to time to get,

$$0 = 2r(t)r'(t)h(t) + r^2(t)h'(t).$$

When $h(t) = 35m$, then $r(t) = 14m$. Putting these in, together with $h'(t) = -13m/min$, we find $r'(t) = (13/5)m/min$.

NAME: STUDENT NO:

Q-5) Let $I_n = \int x^{2n+1} \sin x^2 \ dx$ for integers $n \geq 0$. Find a recursive formula for I_n, and using your formula find explicitly $\int_0^{\sqrt{2\pi}} x^5 \sin x^2 \ dx$.

Solution: Use by-parts with $u = x^{2n}$ to obtain

$$I_n = -(1/2)x^{2n} \cos x^2 + n \int x^{2n-1} \cos x^2 \ dx.$$

For the new integral use again by-parts with $u = x^{2n-2}$ to obtain the final recursive expression

$$I_n = -(1/2)x^{2n} \cos x^2 + (n/2)x^{2n-2} \sin x^2 - n(n-1)I_{n-2}.$$

Clearly $I_0 = -(1/2)\cos x^2 + C$. Putting these together we find that

$$I_2 = -(1/2)x^4 \cos x^2 + x^2 \sin x^2 + \cos x^2 + C,$$

and we then immediately have

$$\int_0^{\sqrt{2\pi}} x^5 \sin x^2 \ dx = -2\pi^2.$$

NAME: STUDENT NO:

Bonus:) A sequence of polynomials, called the Bernoulli polynomials, is defined recursively as follows:

$$P_0(x) = 1, \quad P_n'(x) = nP_{n-1}(x), \quad \text{and} \quad \int_0^1 P_n(x)\, dx = 0.$$

It is known that for example $P_1(x) = x - 1/2$, each $P_n(x)$ is a polynomial in x of degree n with leading coefficient 1, and also that for $n \geq 2$, $P_n(0) = P_n(1)$. Show that $P_n(1-x) = (-1)^n P_n(x)$ for $n \geq 1$.

Solution: Define a function $\phi_n(x) = P_n(x) - (-1)^n P_n(1-x)$ for $n \geq 1$. Check that $\phi_1(x) \equiv 0$. Now assume that $\phi_{n-1}(x) \equiv 0$ for some $n \geq 2$. We want to show that $\phi_n(x) \equiv 0$.

$$\begin{aligned} \phi_n'(x) &= P_n'(x) + (-1)^n P_n'(1-x) \\ &= n[P_{n-1}(x) - (-1)^{n-1} P_{n-1}(1-x)] \\ &= n\phi_{n-1}(x) \equiv 0 \end{aligned}$$

by the induction assumption. Thus we now know that $\phi_n(x)$ is constant. We have to determine what that constant is by finding one particular value of $\phi_n(x)$.

We observe, using the fact that $P_n(0) = P_n(1)$ for $n \geq 2$, that

$$\phi_{n+1}(0) = P_{n+1}(0) - (-1)^{n+1} P_{n+1}(1) = \begin{cases} 2P_{n+1}(0) & \text{if } n+1 \text{ is odd,} \\ 0 & \text{if } n+1 \text{ is even.} \end{cases}$$

$$\phi_{n+1}(1) = P_{n+1}(1) - (-1)^{n+1} P_{n+1}(0) = \begin{cases} 2P_{n+1}(0) & \text{if } n+1 \text{ is odd,} \\ 0 & \text{if } n+1 \text{ is even.} \end{cases}$$

This shows that $\phi_{n+1}(0) = \phi_{n+1}(1)$, so there must exist a point c between 0 and 1 such that $\phi_{n+1}'(c) = 0$. But $\phi_{n+1}'(c) = (n+1)\phi_n(c)$. We found that $\phi_n(c) = 0$, but $\phi_n(x)$ being constant, we conclude that $\phi_n(x) \equiv 0$, and this completes the induction and the proof.
