Date: 4 January 2011, Tuesday NAME:...
Time: 15:30-17:30
Ali Sinan Sertöz STUDENT NO:...

Math 113 Calculus – Final Exam – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

$$\int \sec^3 \theta \, d\theta = \frac{1}{2} \sec\theta \tan\theta + \frac{1}{2} \ln|\sec\theta + \tan\theta| + C$$

Q-1) Let $f : (a, b) \longrightarrow \mathbb{R}$ be a differentiable function. Assume that for some $x_0 \in (a, b)$, $\lim_{x \to x_0} f'(x)$ exists and is L. Show that $f'(x_0) = L$.

Solution: *(This is Question 1 on Homework 1.)*

Assume not. Without loss of generality say $L > f'(x_0)$. Choose an $\epsilon > 0$ with $f'(x_0) < L - \epsilon$. Using the definition of limit, to this $\epsilon > 0$ there corresponds a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$, $x \neq x_0$, we must have $|f'(x) - L| < \epsilon$, in particular $L - \epsilon < f'(x)$.

Now choose any K with $f'(x_0) < K < L - \epsilon$. and any x_1 with $x_0 < x_1 < x_0 + \delta$. On the interval $[x_0, x_1]$ we have $f'(x_0) < K < L - \epsilon < f'(x_1)$. It follows from our interpretation of the limit above that there is no $x \in (x_0, x_1)$ with the property $f'(x) = K$, but this violates the Intermediate Property of the Derivative.

This contradiction proves that we must have $f'(x_0) = L$.

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Q-2) Write your answers to the space provided. No partial credits.

- $f(x) = (\sin x)^x$, $f'(x) = (\sin x)^x \left(\ln \sin x + \dfrac{x \cos x}{\sin x} \right)$.
- $f(x) = (\sqrt{x})^e + (\sqrt{2})^x$, $f'(x) = (e/2)x^{e/2-1} + (\sqrt{2})^x \ln \sqrt{2}$.
- $f(x) = (\ln(\arctan x))^{41}$, $f'(x) = 41\,(\ln(\arctan x))^{40}\, \dfrac{\frac{1}{1+x^2}}{\arctan x}$.
- $f(x) = \displaystyle\int_{x^2}^{\tan x} \sqrt{1+t^3}\,dt$, $f'(x) = (\sqrt{1+\tan^3 x})\,(\sec^2 x) - (\sqrt{1+x^6})\,(2x)$.
- $f(0) = 1,\ f'(0) = 3,\ f(5) = 8,\ f'(5) = 10,\ g(0) = 5,\ g'(0) = 7,\ g(1) = 11,\ g'(1) = 11$

$$\lim_{x\to 0} \frac{g(f(x)) - g(f(0))}{x} = (g \circ f)'\,(0) = g'(f(0))\,f'(0) = g'(1)f'(0) = 11 \cdot 3 = 33.$$

$$\lim_{x\to 0} \frac{f(g(x)) - f(g(0))}{x} = (f \circ g)'\,(0) = f'(g(0))\,g'(0) = f'(5)g'(0) = 10 \cdot 7 = 70.$$

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Q-3) Find $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{n^2+3k^2}\right)/n^2$.

Solution: *(This is almost the same problem as Question 1 in Midterm Exam 2.)*

$$\sum_{k=1}^{n} \left(\sqrt{n^2+3k^2}\right)/n^2 = \sum_{k=1}^{n} \frac{1}{n}\sqrt{1+\left(\frac{\sqrt{3}k}{n}\right)^2}$$

$$= \frac{1}{\sqrt{3}} \sum_{k=1}^{n} \frac{\sqrt{3}}{n}\sqrt{1+\left(\frac{\sqrt{3}k}{n}\right)^2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{n^2+3k^2}\right)/n^2 = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} \sqrt{1+x^2}\, dx$$

$$= \frac{1}{\sqrt{3}} \int_0^{\pi/3} \sec^3\theta \, d\theta$$

$$= \frac{1}{\sqrt{3}} \left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| \Big|_0^{\pi/3} \right)$$

$$= 1 + \frac{1}{2\sqrt{3}} \ln(2+\sqrt{3})$$

$$\approx 1.38.$$

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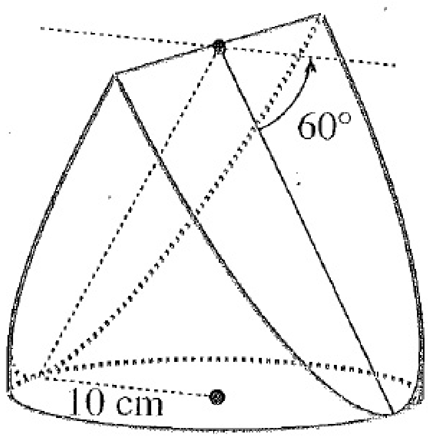
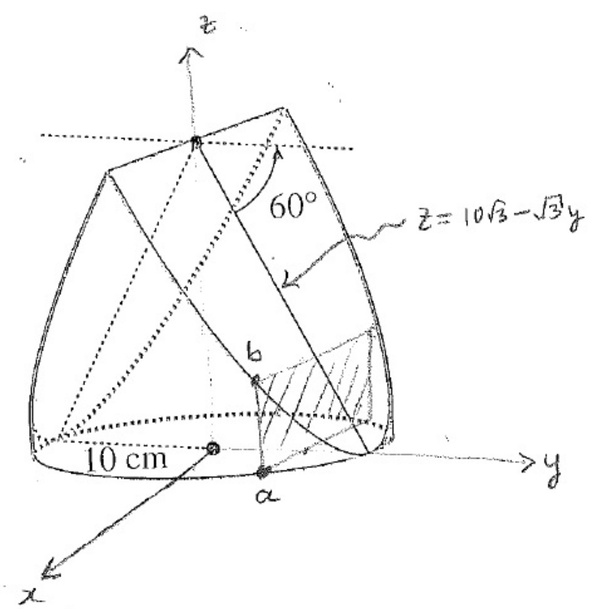
Q-4) The solid in the figure below is cut from a vertical cylinder of radius 10 cm by two planes making angles of 60° with the horizontal. Find its volume.



(This is Exercise 4 on page 454 of your textbook.)

Solution is on next page:

Solution:



The equation of the cylinder is $x^2 + y^2 = 100$. The coordinates of the points a and b are

$$a = (\sqrt{100 - y^2}, y, 0), \quad \text{and} \quad b = (\sqrt{100 - y^2}, y, 10\sqrt{3} - \sqrt{3}y).$$

The area of the shaded rectangle is

$$A(y) = 2\sqrt{100 - y^2}(10\sqrt{3} - \sqrt{3}y).$$

Then volume is

$$\begin{aligned}
V &= 2\int_0^{10} A(y)\,dy \\
&= 4\sqrt{3}\int_0^{10} \sqrt{100 - y^2}\,(10 - y)\,dy \\
&= 40\sqrt{3}\int_0^{10} \sqrt{100 - y^2}\,dy - 4\sqrt{3}\int_0^{10} y\sqrt{100 - y^2}\,dy \\
&= 40\sqrt{3}\left(100\int_0^{\pi/2} \cos^2\theta\,d\theta\right) - 4\sqrt{3}\left(\frac{1}{2}\int_0^{100} u^{1/2}\,du\right) \\
&= 40\sqrt{3}\,(25\pi) - 4\sqrt{3}\left(\frac{1000}{3}\right) \\
&= 1000\sqrt{3}(\pi - \frac{4}{3}) \\
&\approx 3132.
\end{aligned}$$

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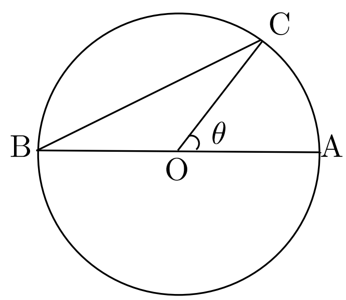
Q-5) Aliye can run twice as fast as she can swim. She is standing at point A on the edge of a circular swimming pool 40 m in diameter, and she wishes to get to the diametrically opposite point B as quickly as possible. She can run around the edge to point C, then swim directly from C to B. Where should C be chosen to minimize the total time taken to get from A to B?



Solution: *(This is Example 5, solved in detail, on page 262 of your textbook.)*

Suppose Aliye swims at the rate k m/sec and hence runs at $2k$ m/sec. If $t = t(\theta)$ is the total time it takes for her to go from A to B via C, then the function to minimize is

$$t(\theta) = \frac{20\theta}{2k} + \frac{40}{k}\sin\frac{\pi-\theta}{2}, \quad \theta \in [0,\pi].$$

For critical points we solve

$$t'(\theta) = \frac{10}{k} - \frac{20}{k}\cos\frac{\pi-\theta}{2} = 0,$$

which gives

$$\cos\frac{\pi-\theta}{2} = \frac{1}{2}, \ \frac{\pi-\theta}{2} = \frac{\pi}{3}, \ \theta = \frac{\pi}{3}.$$

To find the minimum, we evaluate

$$t(0) = \frac{40}{k}, \ t(\frac{\pi}{3}) \approx \frac{45}{k}, \ t(\pi) \approx \frac{31}{k}.$$

Thus for shortest time, C must be situated at B. In other words, to reach to B as quickly as possible, Aliye should run all the way around the pool.