Due Date: October 22, 2010 Friday NAME: ......Zeki Bilir..

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Math 113 Calculus – Homework 1

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Let $f : (a, b) \longrightarrow \mathbb{R}$ be a differentiable function. Assume that for some $x_0 \in (a, b)$, $\lim\limits_{x \to x_0} f'(x)$ exists and is L. Show that $f'(x_0) = L$.

Solution:

Assume not. Without loss of generality say $L > f'(x_0)$. Choose an $\epsilon > 0$ with $f'(x_0) < L - \epsilon$. Using the definition of limit, to this $\epsilon > 0$ there corresponds a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$, $x \neq x_0$, we must have $|f'(x) - L| < \epsilon$, in particular $L - \epsilon < f'(x)$.

Now choose any K with $f'(x_0) < K < L - \epsilon$. and any x_1 with $x_0 < x_1 < x_0 + \delta$. On the interval $[x_0, x_1]$ we have $f'(x_0) < K < L - \epsilon < f'(x_1)$. It follows from our interpretation of the limit above that there is no $x \in (x_0, x_1)$ with the property $f'(x) = K$, but this violates the Intermediate Property of the Derivative.

This contradiction proves that we must have $f'(x_0) = L$.

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Q-2) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Assume that f' is not continuous at some $x_0 \in \mathbb{R}$. Prove or disprove each of the following statements:
(i) It is possible that $\lim_{x \to x_0^+} f'(x) = f'(x_0)$.
(ii) It is possible that $\lim_{x \to x_0^+} f'(x) = L \neq f'(x_0)$.
(iii) It is possible that $\lim_{x \to x_0^+} f'(x) = \infty$.

Solution:

(i) True. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

$$f(x) = \begin{cases} 0 & x \geq 0, \\ x^2 \sin(1/x) & x < 0. \end{cases}$$

When $x > 0$, $f'(x) \equiv 0$.

When $x < 0$, $f'(x) = 2x \sin(1/x) - \cos(1/x)$ and $\lim_{x \to 0^-} f'(x)$ does not exist, so f' is not continuous at the origin.

But we have $\lim_{x \to 0^-} \dfrac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \dfrac{x^2 \sin(1/x)}{x} = \lim_{x \to 0^-} x \sin(1/x) = 0$, and $\lim_{x \to 0^+} \dfrac{f(x) - f(0)}{x} = \lim_{x \to 0^+} 0 = 0$. Hence $f'(0)$ exists and is zero.

Note that we also have $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} 0 = 0 = f'(0)$, so it is possible to have f with the property mentioned in (i).

(ii) and (iii) are false. The solution to problem 1 shows exactly why (ii) cannot be true. For (iii) take L to be any number strictly larger than $f'(x_0)$. Now there exists a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$, $x \neq x_0$, we have $L < f'(x)$. From here on follow the solution of problem 1.

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Q-3) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Assume that $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Prove or disprove the following statement:

There exists a $\delta > 0$ such that f is increasing (strictly or not) on the interval $(x_0 - \delta, x_0 + \delta)$.

Solution:

This must be false. If $f'(x)$ was positive throughout an interval, then f would be increasing on that interval. But the value of f' at only one point does not say much about the behavior of f.

Consider the function

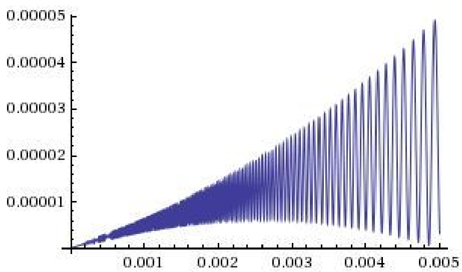
$$f(x) = \begin{cases} x^2 \sin(1/x) + Ax & x \neq 0, \\ 0 & x = 0, \end{cases}$$

for some constant $A > 0$. We can show, as we did in the previous solution, that $f'(x)$ exists everywhere. We can easily show that $f'(0) = A > 0$.

We want to show that for any $\epsilon > 0$, there exist $0 < x_1 < x_2 < \epsilon$ such that $f(x_1) > f(x_2)$ showing that f cannot be increasing in any neighborhood of 0. Let $n > 0$ be an integer such that $x_2 = \dfrac{2}{(4n-1)\pi} < \epsilon$. Then $x_1 = \dfrac{2}{(4n+1)\pi} < x_2 < \epsilon$.

It can be shown that for $A < \dfrac{2}{\pi} = 0.637...$, we will have $f(x_1) > f(x_2)$ as required.

Here is the graph of $y = f(x)$ with $A = 0.005$ and $x \geq 0$:



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Q-4) Find all the points, if any exist, on this ellipse

$$\frac{(x-2)^2}{9}+\frac{(y-3)^2}{4}=1$$

satisfying the property that the line joining the point to the origin is tangent to the ellipse at that point.

(You may use a computer algebra program if need arises.)

Solution:

Explicitly differentiating the equation of the ellipse gives us

$$y'=-\frac{4}{9}\frac{x-2}{y-3}$$

as the slope of the tangent at a point on the ellipse. If the tangent passes through the origin, then its slope must be $\dfrac{y}{x}$. Thus we must solve simultaneously the equations

$$-\frac{4}{9}\frac{x-2}{y-3}=\frac{y}{x} \quad \text{and} \quad \frac{(x-2)^2}{9}+\frac{(y-3)^2}{4}=1.$$

The solutions are

$$(x,y)=\left(\frac{1}{97}(122-27\sqrt{61}),\ \frac{1}{97}(183+8\sqrt{61})\right)\approx(-0.9,\ 2.5)$$

and

$$(x,y)=\left(\frac{1}{97}(122+27\sqrt{61}),\ \frac{1}{97}(183-8\sqrt{61})\right)\approx(3.4,\ 1.2).$$

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Q-5) Find the equation of the tangent line to the curve $x^2y^3 - x^3y^2 = 4$ at the point $(1, 2)$. Show that there is no point $p = (x_0, y_0)$ on the curve where the tangent line to the curve at p passes also from the origin.

Solution:

Implicitly differentiate the given equation and solve for y' to find

$$y' = \frac{3xy - 2y^2}{3xy - 2x^2}.$$

Putting in $(x, y) = (1, 2)$, we find that $y' = -1/2$. Thus the equation of the tangent line is

$$y = -\frac{1}{2}(x - 1) + 2.$$

If the tangent line at (x, y) on the curve passes through the origin, then we must have

$$\frac{3xy - 2y^2}{3xy - 2x^2} = \frac{y}{x}$$

which forces $x = y$ but no such point exists on our curve. Hence no tangent passes through the origin.