Due Date: November 24, 2010 Wednesday NAME:..

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Math 113 Calculus – Homework 2 – Solutions

1	2	3	4	5	TOTAL
20	20	20	20	20	100

Please do not write anything inside the above boxes!

Check that there are 5 questions on your booklet. Write your name on top of every page. Show your work in reasonable detail, unless otherwise stated. A correct answer without proper or too much reasoning may not get any credit.

Q-1) Write the derivatives of the following functions. No partials. Do not show your work.

- $f(x) = x^{3x}, \quad f'(x) = x^{3x}(3\ln x + 3).$
- $f(x) = (\tan x)^{\sec x}, \quad f'(x) = (\tan x)^{\sec x}(\sec x \tan x \ln \tan x + \dfrac{\sec^3 x}{\tan x}).$
- $f(x) = \ln(\cosh x^2), \quad f'(x) = \dfrac{2x \sinh x^2}{\cosh x^2}.$
- $f(x) = x \arctan x^2, \quad f'(x) = \arctan x^2 + \dfrac{2x^2}{1+x^4}.$
- $f(x) = x^{1/\ln x}, \quad f'(\pi) = 0$
- $f(x) = 5^x - x^5, \quad f'(x) = 5^x \ln 5 - 5x^4.$
- $f(x) = x^{\ln x}$, $f'(e) = 2.$
- $f(x) = \dfrac{x^6 - x^4 + 1}{4x^3 + x - 1}$, $f'(0) = -1.$
- Given: $g(0) = 1,\ g(3) = 17,\ g(8) = 0,\ f(0) = 71,\ f(3) = -1,\ f(8) = \sqrt{2},$ $g'(0) = \pi,\ g'(3) = \pi^e,\ g'(8) = e,\ f'(0) = 2^e,\ f'(3) = \ln 3,\ f'(8) = e^{\sqrt{2}}.$

  If $h(x) = f(3g(x) + 5)$, then $h'(0) = 3e^{\sqrt{2}}.$

- Given: $f(5) = \pi/3,\ f'(5) = \pi/4,\ g(5) = 1,\ g'(5) = 0,\ g'(\sqrt{2}/2) = 5,$ $g'(\sqrt{3}/2) = 7,\ g(1/2) = \pi,\ g(\pi/4) = 11.$

  If $h(x) = g(\sin(f(x)))$, then $h'(5) = \dfrac{7\pi}{8}.$

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Q-2) Show that for any $x > -1$ and for any integer $n \geq 0$,

$$(1+x)^n \geq 1 + nx.$$

Solution: We will give two proofs. The first one uses induction.

First of all, the fact that $x > 1$ is necessary so that we don't talk about powers of negative numbers which are sometimes imaginary.

Clearly the statement is true for $n = 0$. Assume that it is true for n and check what happens for $n+1$.

$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x)$ by induction hypothesis and because $1 + x > 0$.

But we also have

$$(1+nx)(1+x) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x \text{ since } nx^2 \geq 0.$$

This then shows that the statement holds for $n+1$ when it holds for n, completing the proof.

For the second proof, observe that the statement is clearly true for $n = 0$ and $n = 1$. So assume $n \geq 2$ and consider the function

$$f(x) = (1+x)^n - (1+nx) \text{ for } x \geq 1.$$

We check the derivative of this function.

$$f'(x) = n(1+x)^{n-1} - n$$

which is negative for $-1 \leq x < 0$, positive for $x > 0$ and zero for $x = 0$. Since $f(-1) = n - 1 > 0$ and $f(0) = 0$, we conclude that $f(x) \geq 0$ for all $x \leq -1$.

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Q-3) Sketch the graph of $f(x) = \dfrac{x+1}{x^2+1}\cdot$ Find the absolute minimum and maximum values of f.

Solution: We first observe what we can without using the derivative.

$\lim_{x\to\pm\infty} f(x) = 0$. $f(x) < 0$ for $x < -1$. $f(x) > 0$ for $x > -1$. $f(-1) = 0$. $f(0) = 1$.

Next we check the derivative. $f'(x) = -\dfrac{(x-\alpha_1)(x-\alpha_2)}{(x^2+1)^2}$, where
$\alpha_1 = -1+\sqrt{2} \approx 0.4$ and $\alpha_2 = -1-\sqrt{2} \approx -2.4$.

Then we look at the second derivative. $f''(x) = \dfrac{2(x-1)(x-\beta_1)(x-\beta_2)}{(x^2+1)^3}$, where
$\beta_1 = -2+\sqrt{3} \approx 0.26$ and $\beta_2 = -2-\sqrt{3} \approx -3.7$.

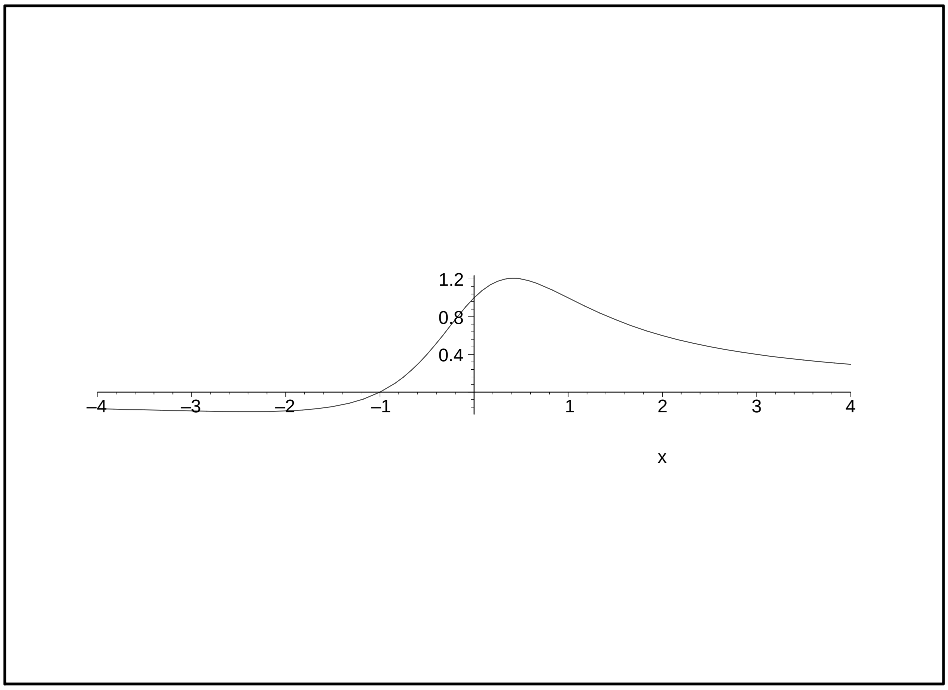
Putting these data together in a comparison table, we obtain the sketch of the graph.

The minimum value is $f(\alpha_2) \approx -0.2$ and the maximum value is $f(\alpha_1) \approx 1.2$.

Here is the table:

	−∞	−3.7	−2.4	−1	−0.2	0	0.4	1 +∞
f	0			0	1			0
f′	−	− 0	+	+	+	+ 0	−	−
f″	− 0	+	+	+ 0	−	−	− 0	+
	⌢	⌣	⌣	⌣	⌢	⌢	⌢	⌣
	↘	↘	↗	↗	↗	↗	↘	↘

And here is the graph:



NAME: STUDENT NO:

Q-4) Sketch the graph of $f(x) = x^2 e^{-x^2}$. Find the absolute minimum and maximum values of f.

Solution: Check that $\lim_{x \to \pm\infty} f(x) = 0$.

Next $f'(x) = -2x(x-1)e^{-x^2}$, and $f''(x) = 2(2x^4 - 5x^2 + 1)e^{-x^2}$.

The first derivative vanishes at $x = 0$, and $x = \pm 1$.

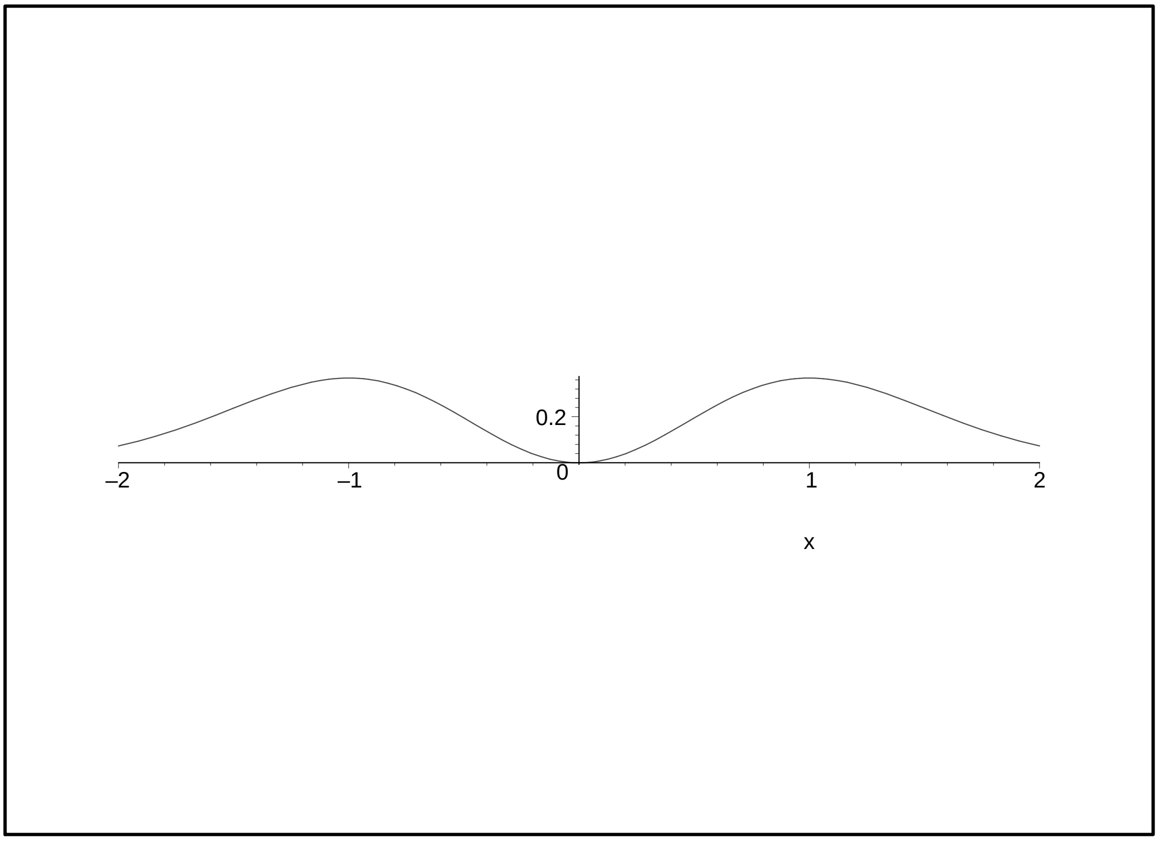
The second derivative vanishes at $x = \pm\alpha$ and $x = \pm\beta$ where
$\alpha = \dfrac{\sqrt{5+\sqrt{17}}}{2} \approx 1.5$ and $\beta = \dfrac{\sqrt{5-\sqrt{17}}}{2} \approx 0.4$.

The minimum value is $f(0) = 0$ and the maximum value is $f(\pm 1) = 1/e \approx 0.36$.

Here is the table:

	$-\infty$		-1.5		-1		-0.4		0		0.4		1		1.5		$+\infty$
f	0								0								0
f′		+		+	0	−		−	0	+		+	0	−		−	
f″		+	0	−		−	0	+		+	0	−		−	0	+	
		∪		∩		∩		∪		∪		∩		∩		∪	
		↗		↗		↘		↘		↗		↗		↘		↘	

And here is the graph:



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Q-5) Approximate $\tan 1$ with an absolute error less than $1/1000$, using the Taylor polynomials of $\sin x$ and $\cos x$.

Solution: We try the Taylor polynomials of $\sin x$ and $\cos x$ at $x = 1$. The number of terms needed is determined by trial and error.

First observe that

$$A = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} < \cos 1 < 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \frac{1}{8!} = B$$

and

$$C = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} < \sin 1 < 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} = D.$$

Then we clearly have

$$1.557401 \approx \frac{C}{B} < \tan 1 < \frac{D}{A} \approx 1.557478.$$

Since $\dfrac{D}{A} - \dfrac{C}{B} \approx 0.00007$, we can take as $\tan 1$ the value

$$\tan 1 \approx \frac{1}{2}\left(\frac{D}{A} + \frac{C}{B}\right) \approx 1.557440.$$

It would require around 25 terms from the Taylor expansion of $\tan x$ to find such an approximation and even then we would have a terrible time in controlling the error.

Send comments to sertoz@bilkent.edu.tr