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Math 113 Calculus – Homework 4 and 5 – Solutions

The following problems are taken from your text book.

HWK-4)

$*$**31.** A wine glass in the shape of a right-circular cone of height h and semivertical angle α (see Figure 7.14) is filled with wine. Slowly a ball is lowered into the glass, displacing wine and causing it to overflow. Find the radius R of the ball that causes the greatest volume of wine to overflow out of the glass.

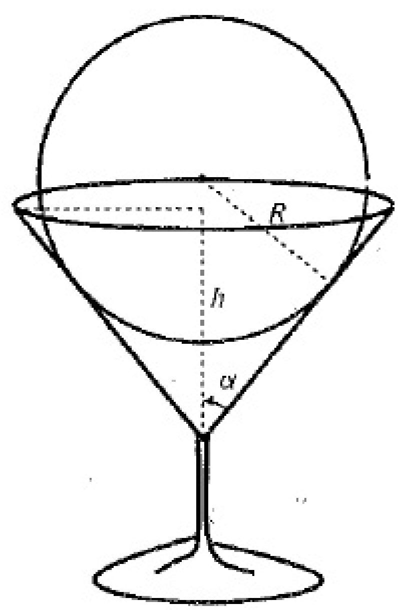
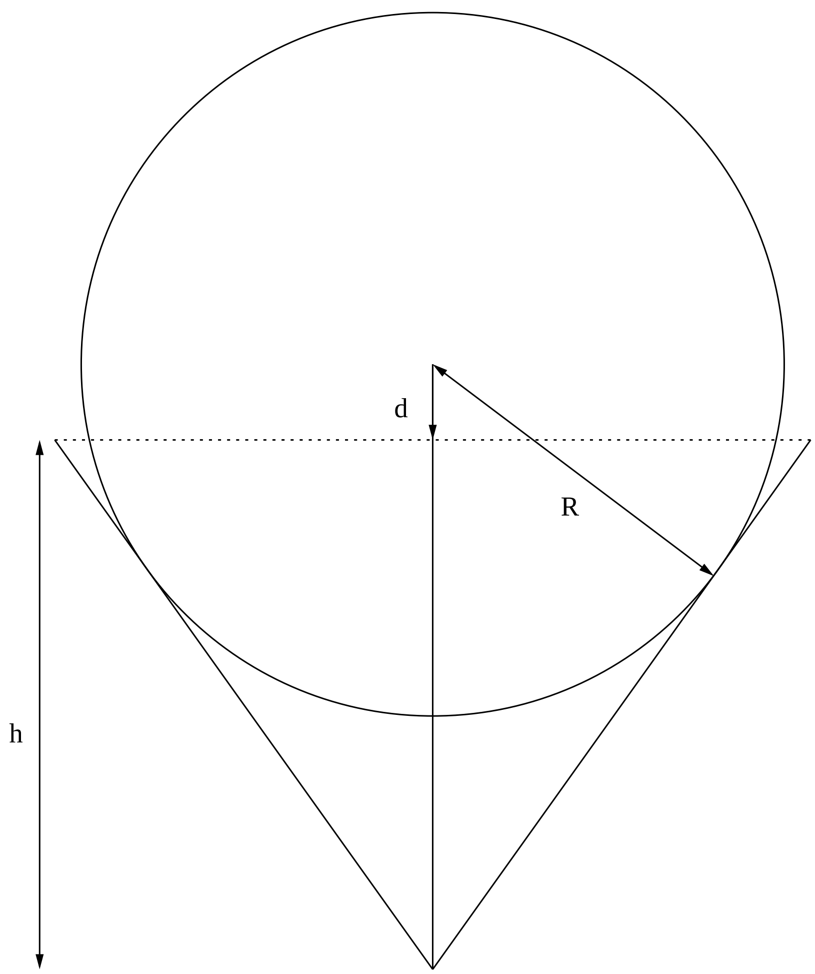


Figure 7.14

Solution: First we write a function which calculates the volume of a cap cut off from a sphere of radius R by a plane d distance away from its center, see figure below. Such a plane cuts the sphere into two caps and to distinguish between them we define a positive direction away from the center of the ball and choose the cap that contains the point $d = R > 0$ distance away from the center. Call this volume $V(R, d)$. This is a volume of revolution and is calculated as follows.

$$\begin{aligned} V(R,d) &= \pi \int_d^R r^2(t)\,dt \\ &= \pi \int_d^R (R^2 - t^2)\,dt \\ &= \pi \left(R^2 t - \frac{1}{3} t^3 \Big|_d^R \right) \\ &= \pi (\frac{2}{3} R^3 - R^2 d + \frac{1}{3} d^3). \end{aligned}$$



Next we return to our wine glass, again using the above figure. Assume first that the center of the sphere remains above wine level after it is dropped into the glass. Call the distance between the level of wine and the center of ball d. We see that

$$d = \frac{R}{\sin\alpha} - h.$$

As R increases, d increases. Also observe that when R is small, then the center of the ball is below the wine level and in that case d will be negative. The above algebraic expression for d agrees with this. Therefore we can then take the lower limit of R as zero and express d as above for $R \geq 0$.

For the upper limit of R we may take that value of R which makes the ball tangent to the sides of the glass at the top end of the glass. Any larger ball will spill less wine. However our algebraic expression for the spilled wine assumes that the ball is always tangent to the *extended* sides of the glass and the expression continues to make sense as R increases until the ball is so big that while it is tangent to the extended sides of the glass it is also tangent to the surface of the wine. This value of R is,

$$R = \frac{h\sin\alpha}{1-\sin\alpha}$$

and we take it as the upper limit of R. The amount of spilled water is zero in this case.

If we denote by $f(R)$ the amount of spilled wine by a ball of radius R, then this function is define for

$$0 \leq R \leq \frac{h\sin\alpha}{1-\sin\alpha}$$

and the function is zero at both end points.

The expression for $f(R)$ is

$$\begin{aligned}
f(R) &= V(R,d)\\
&= V(R, \frac{R}{\sin\alpha} - h)\\
&= \frac{2}{3}R^3 - R^2(\frac{R}{\sin\alpha} - h) + \frac{1}{3}(\frac{R}{\sin\alpha} - h)^3\\
f'(R) &= \left(2 - \frac{3}{\sin\alpha} + \frac{1}{\sin^3\alpha}\right)R^2 + \left(2h - \frac{2h}{\sin^2\alpha}\right) + \frac{h^2}{\sin\alpha}
\end{aligned}$$

and equating $f'(R)$ to zero we find

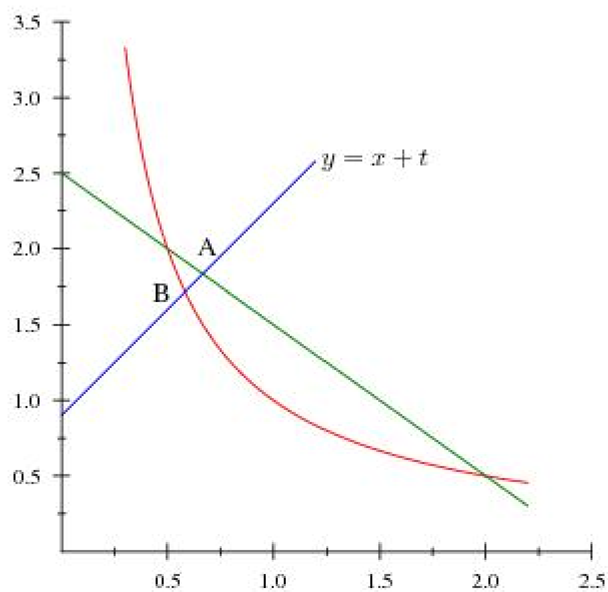
$$R_1 = \frac{h\sin\alpha}{1-\sin\alpha},\ R_2 = \frac{h\sin\alpha}{1+\sin\alpha - 2\sin^2\alpha} = \frac{h\sin\alpha}{\sin\alpha + \cos 2\alpha}.$$

The value of R_1 corresponds to the upper end point for R and in fact as expected we calculate to see that $f(R_1) = 0$.

We see that $0 < R_2 < R_1$ for $0 < \alpha < \pi/2$. Hence $R = R_2$ gives the maximum spill. Note further that the center of the ball giving the maximum spill will be below wine level for $0 < \alpha < \pi/6$, and above wine level for $\pi/6 < \alpha < \pi/2$. Clearly the problem does not make sense for $\alpha = 0$ and $\alpha = \pi/2$.

HWK-5)

$*$**32.** The finite plane region bounded by the curve $xy = 1$ and the straight line $2x + 2y = 5$ is rotated about that line to generate a solid of revolution. Find the volume of that solid.



Solution: The curve $y = 1/x$ and the line $y = 5/2 - x$ intersect at the points $(1/2, 2)$ and $(2, 1/2)$. For any t, the line $y = x + t$ is at right angles with the line $y = 5/2 - x$. For the line $y = x + t$ to intersect the region of revolution, it must intersect the y-axis between $-3/2$ and $3/2$. Let the line $y = x + t$ intersect the $y = 5/2 - x$ line at A, and the curve $y = 1/x$ at B. The coordinates of A and B can be calculated to be

$$A = \left(\frac{5-2t}{4}, \frac{5+2t}{4}\right), \text{ and } B = \left(\frac{\sqrt{t^2+4}-t}{2}, \frac{\sqrt{t^2+4}+t}{2}\right).$$

From these we find

$$\begin{aligned} |AB|^2 &= \left(\frac{5-2t}{4} - \frac{\sqrt{t^2+4}-t}{2}\right)^2 + \left(\frac{5+2t}{4} - \frac{\sqrt{t^2+4}+t}{2}\right)^2 \\ &= \frac{1}{8}\left(5 - 2\sqrt{t^2+4}\right)^2 \\ &= \frac{1}{8}\left(41 - 20\sqrt{t^2+4} + 4t^2\right). \end{aligned}$$

We revolve the cylindrical shell $|AB|\Delta s$ around the line $y = 5/2 - x$ where Δs is length along that line. However $\Delta s = \Delta t/\sqrt{2}$ where t takes values along the y-axis. Therefore the required volume is

$$\begin{aligned} Volume &= \frac{\pi}{\sqrt{2}} \int_{-3/2}^{3/2} |AB|^2 \, dt \\ &= \frac{\pi}{8\sqrt{2}} \int_{-3/2}^{3/2} \left(41 - 20\sqrt{t^2+4} + 4t^2\right) \, dt \\ &= \frac{\pi}{8\sqrt{2}} \left(41t - 20 \left(\sinh^{-1}\left(\frac{t}{2}\right) + \frac{t\sqrt{t^2+4}}{2} \right) + \frac{4}{3}t^3 \Bigg|_{-3/2}^{3/2} \right) \\ &= \frac{\pi}{8\sqrt{2}} \left(57 - \sinh^{-1}\left(\frac{3}{4}\right) \right) \\ &\approx 0.4299. \end{aligned}$$